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Abstract

The optimal weights on indicators in models with partial information about the state of the economy and forward-looking variables are derived and interpreted, both for equilibria under discretion and under commitment. An example of optimal monetary policy with a partially observable potential output and a forward-looking indicator is examined. The optimal response to the *optimal estimate* of potential output displays certainty-equivalence, whereas the optimal response to the *imperfect observation* of output depends on the noise in this observation.

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Contents

1 Introduction

It is a truism that monetary policy operates under considerable uncertainty about the state of the economy and the size and nature of the disturbances that hit the economy. This is a particular problem for a procedure such as inflation-forecast targeting, under which a central bank, in order to set its interest-rate instrument, needs to construct conditional forecasts of future inflation, conditional on alternative interest-rate paths and the bank's best estimate of the current state of the economy and the likely future development of important exogenous variables.[1] Often, different indicators provide conflicting information on developments in the economy. In order to be successful, a central bank then needs to put the appropriate weights on different information and draw the most efficient inference. In the case of a purely backward-looking model (both of the evolution of the bank's target variables and of the indicators), the principles for efficient estimation and signal extraction are well known. But in the more realistic case where important indicator variables are forward-looking variables, the problem of efficient signal-extraction is inherently more complicated. The purpose of this paper is to clarify the principles for determining the optimal weights on different indicators in such an environment.

In the case where there are no forward-looking variables, it is well known that a linear model with a quadratic loss function and a partially observable state of the economy (partial information) is characterized by certainty-equivalence. That is, the optimal policy is the same as if the state of the economy were fully observable (full information), except that one responds to an efficient estimate of the state vector rather than to its actual value. Thus, a separation principle applies, according to which the selection of the optimal policy (the optimization problem) and the estimation of the current state of the economy (the estimation or signal-extraction problem) can be treated as separate problems. In particular, the observable variables will be predetermined and the innovations in the observable variables (the difference between the current realization and previous prediction

[1] See Svensson [27], [29] and [32] for discussion of inflation targeting and references to the literature.

of each of the observable variables) contain all new information. The optimal weights to be placed on the innovations in the various observable variables in one's estimate of the state vector at each point in time are provided by a standard Kalman filter (see, for instance, Chow [3], Kalchenbrenner and Tinsley [14] and LeRoy and Waud [15]).[2]

The case without forward-looking variables is, however, very restrictive. In the real world, many important indicator variables for central banks are forward-looking variables, variables that depend on private-sector expectations of the future developments in the economy and future policy. Central banks routinely watch variables that are inherently forward-looking, like exchange rates, bond rates and other asset prices, as well as measures of private-sector inflation expectations, industry order-flows, confidence measures, and the like. Forward-looking variables complicate the estimation or signal-extraction problem significantly. They depend, by definition, on private-sector expectations of future endogenous variables and of current and future policy actions. However, these expectations in turn depend on an estimate of the current state of the economy, and that estimate in turn depends, to some extent, on observations of the current forward-looking variables. This circularity presents a considerable challenge for the estimation problem in the presence of forward-looking variables.

It is well known that forward-looking variables also complicate the optimization problem. For example, optimal policy under commitment ceases in general to coincide with the outcome of discretionary optimization, as demonstrated for the general linear model with quadratic objectives in Backus and Driffill [2] and Currie and Levine [6]. With regard to the estimation problem, Pearlman, Currie and Levin [20] showed in a linear (non-optimizing) model with forward-looking variables and partial symmetric information that the solution can be expressed in terms of a Kalman filter, although the solution is much more complex than in the purely backward-looking case. Pearlman [19] later used this solution in an optimizing model to demonstrate that certainty-equivalence, and hence the sepa-

[2] See Gerlach and Smets [10], Peersman and Smets [22] and Smets [24] for recent applications to estimation of the output gap in purely backward-looking frameworks.

ration principle, applies under both discretion and commitment, in the presence of forward-looking variables and symmetric partial information.

The present paper extends this previous work on partial information with forward-looking variables by providing simpler derivations of the optimal weights on the observable variables, and clarifying how the updating equations can be modified to handle the circularity mentioned above. We also provide a simple application, in a now-standard model of monetary policy with a forward-looking aggregate supply relation and a forward-looking "expectational IS" relation.

Section 2 presents a relatively general linear model of an aggregate private sector and a policy-maker, called the central bank, with a quadratic loss function. It then characterizes optimizing policy under discretion, demonstrates certainty-equivalence, and derives the corresponding updating equation in the Kalman filter for the estimation problem. Section 3 does the same for the optimal policy with commitment.[3] Throughout the paper, we maintain the assumption of symmetric information between the private-sector and the central bank; the asymmetric case where certainty-equivalence does not hold is treated in Svensson and Woodford [36].

Section 4 discusses the interpretation of the Kalman filter. It shows how the Kalman filter can be modified to handle the simultaneity and circularity referred to above, and that the current estimate of the state of the economy can be expressed as a distributed lag of current and past observable variables, with the Kalman gain matrix providing the optimal weights on the observable variables. Section 5 presents an example of optimal monetary policy in a simple forward-looking model, where inflation is forward-looking and depends on expectations of future inflation, on a partially observable output gap (the difference between observable output and a partially unobservable potential output), and on an unobservable "cost-push" shock. Since the observable rate of inflation both affects and depends on the current estimates of potential output and the cost-push shock, this example illustrates the gist of the estimation problem with forward-looking

[3] The demonstration of certainty-equivalence under commitment raises some special difficulties which are treated in a separate paper, Svensson and Woodford [37].

variables. Finally, section 6 presents some conclusions, while Appendices A–D report some technical details.

2 Optimization under discretion

We consider a linear model of an economy with two agents, an (aggregate) private sector and a policymaker, called the central bank. The model is given by

$$\begin{bmatrix} X_{t+1} \\ Ex_{t+1|t} \end{bmatrix} = A^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + A^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + Bi_t + \begin{bmatrix} u_{t+1} \\ 0 \end{bmatrix}, \tag{2.1}$$

where X_t is a vector of n_X *predetermined variables* in period t, x_t is a vector of n_x *forward-looking variables*, i_t is (a vector of) the central bank's n_i policy *instrument*(*s*), u_t is a vector of n_X iid shocks with mean zero and covariance matrix Σ_{uu}, and A^1, A^2, B and E are matrices of appropriate dimension. The $n_x \times n_x$ matrix E (which should not be confused with the expectations operator $\mathrm{E}[\cdot]$) may be singular (this is a slight generalization of usual formulations when E is the identity matrix). For any variable z_t, $z_{\tau|t}$ denotes $\mathrm{E}[z_\tau | I_t]$, the rational expectation (the best estimate) of z_τ given the information I_t, the information available in period t to the central bank. The information is further specified below. Let Y_t denote a vector of n_Y *target variables* given by

$$Y_t = C^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + C^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + C_i i_t, \tag{2.2}$$

where C^1, C^2 and C_i are matrices of appropriate dimension. Let the quadratic form

$$L_t = Y_t' W Y_t \tag{2.3}$$

be the central bank's period loss function, where W is a positive-semidefinite weight matrix.

Let the vector of n_Z *observable variables*, Z_t, be given by

$$Z_t = D^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + D^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + v_t, \tag{2.4}$$

where v_t, the vector of noise, is iid with mean zero and covariance matrix Σ_{vv}. The information I_t in period t is given by

$$I_t = \{Z_\tau, \tau \leq t;\ A^1, A^2, B, C^1, C^2, C_i, D^1, D^2, E, W, \delta, \Sigma_{uu}, \Sigma_{vv}\}, \tag{2.5}$$

where δ $(0 < \delta < 1)$ is a discount factor (to be introduced below). This incorporates the case when some or all of the predetermined and forward-looking variables are observable.[4]

Note that (2.1) assumes that the expectations $x_{t+1|t}$ in the second block of equations are conditional on the information I_t. This corresponds to the case when the private sector and the central bank has the same information I_t, so information is assumed to be symmetric. The case of asymmetric information when these expectations are replaced by a private sector expectations $\mathrm{E}[x_{t+1}|I_t^p]$ where the private-sector information I_t^p differs from I_t is treated in Svensson and Woodford [36].

Assume first that there is no commitment mechanism, so the central bank acts under discretion. Assume that central bank each period, conditional on the information I_t, minimizes the expected discounted current and future values of the intertemporal loss function,

$$\mathrm{E}[\sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau}|I_t]. \tag{2.6}$$

As shown in Pearlman [19] and in appendix A, *certainty-equivalence* applies when the central bank and the private sector has the same information. Certainty-equivalence means that the *estimation* of the partially observed state of the economy can be separated from the *optimization*, the setting of the instrument so as to minimize the intertemporal loss function.

The equilibrium under discretion will be characterized by the instrument being a linear function of the current estimate of the predetermined variables,

$$i_t = F X_{t|t}. \tag{2.7}$$

[4] Note that the predetermined and forward-looking variables can be interpreted as deviations from unconditional means and the target variables can be interpreted as deviations from constant target levels. More generally, constants, non-zero unconditional means and non-zero target levels can be incorporated by including unity among the predetermined variables, for instance, as the last element of X_t. The last row of the relevant matrices will then include the corresponding constants/means/target levels.

Furthermore, the estimate of the forward-looking variables will fulfill

$$x_{t|t} = GX_{t|t}, \tag{2.8}$$

where the matrix G by appendix A fulfills

$$G = (A_{22} - EGA_{12})^{-1}[-A_{21} + EGA_{11} + (EGB_1 - B_2)F], \tag{2.9}$$

where

$$A \equiv A^1 + A^2, \tag{2.10}$$

the matrices A, A^j ($j = 1, 2$) and B are decomposed according to X_t and x_t,

$$A^j = \begin{bmatrix} A^j_{11} & A^j_{12} \\ A^j_{21} & A^j_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and we assume that the matrix $A_{22} - EGA_{12}$ is invertible. The matrices F and G depend on A, B, $C \equiv C^1 + C^2$, C_i, E, W and δ, but (corresponding to the certainty-equivalence referred to above) *not* on D^1, D^2, Σ_{uu} and Σ_{vv}.

Now, the lower block of (2.1) implies

$$A^1_{21}(X_t - X_{t|t}) + A^1_{22}(x_t - x_{t|t}) = 0. \tag{2.11}$$

Combining this with (2.8) and assuming that A^1_{22} is invertible gives

$$x_t = G^1 X_t + G^2 X_{t|t}, \tag{2.12}$$

where G^1 and G^2 fulfill

$$G^1 = -(A^1_{22})^{-1} A^1_{21}, \tag{2.13}$$

$$G^2 = G - G^1. \tag{2.14}$$

The matrices G^1 and G^2 depend on G and A^1, hence also on B, $C \equiv C^1 + C^2$, C_i, E, W and δ, but (because of the certainty-equivalence) they are independent of D^1, D^2, Σ_{uu} and Σ_{vv}.

It follows from (2.7) and (2.12) that the dynamics for X_t and Z_t follows

$$X_{t+1} = HX_t + JX_{t|t} + u_{t+1}, \tag{2.15}$$

$$Z_t = LX_t + MX_{t|t} + v_t, \tag{2.16}$$

where

$$H \equiv A_{11}^1 + A_{12}^1 G^1, \tag{2.17}$$

$$J \equiv B_1 F + A_{12}^1 G^2 + A_{11}^2 + A_{12}^2 G, \tag{2.18}$$

$$L \equiv D_1^1 + D_2^1 G^1, \tag{2.19}$$

$$M \equiv D_2^1 G^2 + D_1^2 + D_2^2 G, \tag{2.20}$$

where $D^j = [D_1^j \;\; D_2^j]$ $(j = 1, 2)$ is decomposed according to X_t and x_t. (Note that the matrix L in (2.19) should not be confused with the period loss function L_t in (2.3).)

We note that the problem of estimating the predetermined variables has been transformed to a problem without forward-looking variables, (2.15) and (2.16). This means that the estimation problem becomes a simpler variant of the estimation problem with forward-looking variables that is solved in Pearlman, Currie and Levine [20]. The derivations below is hence a simplification of that in [20].[5]

2.1 Optimal filtering

Assume that the optimal prediction of X_t will be given by a Kalman filter,

$$X_{t|t} = X_{t|t-1} + K(Z_t - LX_{t|t-1} - MX_{t|t}), \tag{2.21}$$

where the Kalman gain matrix K remains to be determined. We can rationalize (2.21) by observing that $Z_t - MX_{t|t} = LX_t + v_t$, hence,

$$Z_t - LX_{t|t-1} - MX_{t|t} = L(X_t - X_{t|t-1}) + v_t,$$

so (2.21) can be written in the conventional form

$$X_{t|t} = X_{t|t-1} + K[L(X_t - X_{t|t-1}) + v_t], \tag{2.22}$$

which allows us to identify K as (one form of) the Kalman gain matrix.[6] From (2.15) we get

$$X_{t+1|t} = (H + J)X_{t|t}, \tag{2.23}$$

[5] Pearlman [19] refers to the complex derivation of the Kalman filter in Pearlman, Currie and Levine [20] but doesn't report that the derivation is actually much easier than in [20].

[6] Harvey [12] definies the Kalman gain matrix in this way, whereas Harvey [13] defines it as the transition matrix (yet to be specified in our case) times K.

and the dynamics of the model are given by (2.15), (2.12), (2.22) and (2.23).

It remains to find an expression for K. Appendix B shows, by expressing the problem in terms of the prediction errors $X_t - X_{t|t-1}$ and $Z_t - Z_{t|t-1}$, that K is given by

$$K = PL'(LPL' + \Sigma_{vv})^{-1}, \tag{2.24}$$

where the matrix $P \equiv \mathrm{Cov}[X_t - X_{t|t-1}]$ is the covariance matrix for the prediction errors $X_t - X_{t|t-1}$ and fulfills

$$P = H[P - PL'(LPL' + \Sigma_{vv})^{-1}LP]H' + \Sigma_{uu}. \tag{2.25}$$

Thus P can be solved from (2.25), either numerically or analytically, depending upon the complexity of the matrices H, L and Σ_{uu}. Then K is given by (2.24).

Note that (2.24) and (2.25) imply that K only depends on A^1, D^1, Σ_{uu} and Σ_{vv}, and hence is independent of C^1, C^2, C_i, W and δ. Thus, K is independent of the policy chosen. This demonstrates that the determination of the optimal policy given an estimate of the state of the economy and the estimation of the state of the economy can be treated as separate problems, as in the case without forward-looking variables treated in Chow [3], Kalchenbrenner and Tinsley [14] and LeRoy and Waud [15]. This is no longer true under asymmetric information, as demonstrated in Svensson and Woodford [36].

3 Optimal policy with commitment

Consider again the model described by equations (2.1)–(2.4), but suppose instead that the central bank commits itself in an initial *ex ante* state (prior to the realization of any period zero random variables) to a state-contingent plan for the indefinite future that minimizes the expected discounted losses

$$\mathrm{E}\left[\sum_{t=t_0}^{\infty} \delta^t L_t\right].$$

Here $\mathrm{E}[\cdot]$ indicates the expectation with respect to information in the initial state in period t_0, in which the commitment is made. It is important to consider optimal commitment from such an *ex ante* perspective, because, in the case of partial information, the information that the central bank possesses in any given state depends upon the way that it has committed itself to behave in other states that might have occurred instead.

As shown in Pearlman [19] for a slightly less general case, certainty-equivalence applies in this case as well. A more intuitive proof of certainty-equivalence is supplied in Svensson and Woodford [37]. Svensson and Woodford [37] show that the optimal policy under commitment satisfies

$$i_t = FX_{t|t} + \Phi\,\Xi_{t-1}, \tag{3.1}$$

$$x_{t|t} = GX_{t|t} + \Gamma\,\Xi_{t-1}, \tag{3.2}$$

$$\Xi_t = SX_{t|t} + \Sigma\,\Xi_{t-1}, \tag{3.3}$$

for $t \geq t_0$, where F, G, S, Φ, Γ and Σ are matrices of appropriate dimension, and Ξ_t is the vector of the n_x Lagrange multiplier of the lower block of (2.1), the equations corresponding to the forward-looking variables. Furthermore, $\Xi_{t_0-1} = 0$.

Woodford [42] and Svensson and Woodford [35] discuss a socially optimal equilibrium in a "timeless perspective," which involves a stationary equilibrium corresponding to a commitment made far in the past, corresponding to $t_0 \to -\infty$. Then, (3.1)–(3.3) apply for all $t > -\infty$. Here, we consider this stationary equilibrium.

Note that (3.3) can then be solved backward to yield

$$\Xi_{t-1} = \sum_{\tau=0}^{\infty} \Sigma^{\tau} S X_{t-1-\tau|t-1-\tau}.$$

Thus, the most fundamental difference with respect to the discretion case is that, under the optimal commitment, $x_{t|t}$ is no longer a linear function of the current estimate of the predetermined variable alone, $X_{t|t}$, but instead depends upon past estimates $X_{t-\tau|t-\tau}$ as well. The inertial character of optimal policy that this can result in is illustrated in Woodford [41] and [42] and in Svensson and Woodford [35].

Svensson and Woodford [35] also show that the socially optimal equilibrium can be achieved under discretion, if the intertemporal loss function in period t is modified to equal

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau} + \Xi_{t-1}(x_t - x_{t|t-1}). \tag{3.4}$$

That is, the central bank internalizes the cost of letting the forward-looking variables, x_t, deviate from previous expectations, $x_{t|t-1}$, using the Lagrange multiplier Ξ_{t-1} for (5.1) in period $t-1$, thus determined in the previous period, as a measure of that cost.[7]

As explained in detail in Svensson and Woodford [37], the matrices F, G, S, Φ, Γ and Σ depend on A, B, C, C_i, W and δ, but that they are independent of Σ_{uu}. Thus, these coefficients are the same as in the optimal plan under *certainty*. This is the *certainty-equivalence* result for the case of partial information.

Using the same reasoning as in the derivation of (2.12) and substituting in (3.2) for $x_{t|t}$, we obtain

$$x_t = G^1 X_t + G^2 X_{t|t} + \Gamma\, \Xi_{t-1}, \tag{3.5}$$

where G^1 and G^2 again are given by (2.13) and (2.14). Again, the matrices G^1 and G^2, like the others, are independent of the specifications of D, Σ_{uu}, and Σ_{vv}.

Substitution of (3.1), (3.2) and (3.5) into the first row of (2.1) furthermore yields

$$X_{t+1} = HX_t + JX_{t|t} + \Psi\, \Xi_{t-1} + u_{t+1}, \tag{3.6}$$

where H and J are again given by (2.17) and (2.18), and

$$\Psi \equiv A_{12}\Gamma + B_1\Phi. \tag{3.7}$$

Equations (3.3) and (3.5)–(3.6) then describe the evolution of the predetermined and forward-looking variables, X_t and x_t, once we determine the evolution of the estimates $X_{t|t}$ of the predetermined variables.

3.1 Optimal filtering

Substituting (3.5) into (2.4), we obtain

$$Z_t = LX_t + MX_{t|t} + \Lambda\, \Xi_{t-1} + v_t, \tag{3.8}$$

[7] Adding a linear term to the loss function is similar to the linear inflation contracts discussed in Walsh [39] and Persson and Tabellini [23]. Indeed, the term added in (3.4) corresponds to a state-contingent linear inflation contract, which, as discussed in Svensson [28], can remedy both stabilization bias and average-inflation bias.

where L and M are again given by (2.19) and (2.20), and

$$\Lambda \equiv D_2 \Gamma. \tag{3.9}$$

Equations (3.6) and (3.8) are then the transition and measurement equations for an optimal filtering problem. Again the transformation into a problem without forward-looking variables allows us to derive the estimation equations in a manner that is simpler than that used in Pearlman, Currie and Levine [20].

The optimal linear prediction of X_t is again given by a Kalman filter,

$$X_{t|t} = X_{t|t-1} + K(Z_t - LX_{t|t-1} - MX_{t|t} - \Lambda\, \Xi_{t-1}), \tag{3.10}$$

analogously to (2.21). From (3.6) we get

$$X_{t+1|t} = (H + J)X_{t|t} + \Psi\, \Xi_{t-1}, \tag{3.11}$$

and a complete system of dynamic equations for the model is then given by (3.3), (3.5), (3.6), (3.10) and (3.11).

It remains to find an expression for the Kalman gain matrix K. Again, as in appendix B, it is practical to work in terms of the prediction errors $X_t - X_{t|t-1}$ and $Z_t - Z_{t|t-1}$, and equations (B.1)–(B.13) and (2.24)–(2.25) continue to apply, exactly as in the discretion case. Note that this implies that the Kalman gain matrix K is exactly the same matrix as in the discretion equilibrium; in fact, it depends only upon the matrices A^1, Σ_{uu}, D^1 and Σ_{vv}.

4 Optimal weights on indicators: General remarks

In this section, we offer some general conclusions about the way in which the vector of observed variables Z_t, the indicators, is used to estimate the current state of the economy. As in sections 2 and 3, we assume that the central bank and the private sector have the same information, but our comments apply both to the discretion equilibrium and the commitment equilibrium. In either case, the observed variables matter only insofar as they affect the central bank's estimate $X_{t|t}$ of the predetermined states.

Let us restate (2.4) and (3.8),

$$\begin{aligned} Z_t &= D_1^1 X_t + D_2^1 x_t + D_1^2 X_{t|t} + D_2^2 x_{t|t} + v_t \\ &= L X_t + M X_{t|t} + \Lambda\, \Xi_{t-1} + v_t, \end{aligned}$$

where we note that the second equation applies also in the discretion case, if we set $\Lambda \equiv 0$ in that case. When $D_2^1 \neq 0$, the observable variables include or depend on the forward-looking variables. Then there is a contemporaneous effect of $X_{t|t}$ on Z_t, due to the effect of $X_{t|t}$ on both expectations $x_{t+1|t}$ and the equilibrium choice of the instrument i_t. If $D_1^2 \neq 0$, there is a direct effect of $X_{t|t}$ on the observable variables; if $D_2^2 \neq 0$, there is an effect of $X_{t|t}$ on the observable variables via $x_{t|t}$. In the commitment case, if $\Lambda \neq 0$, there is also a lagged effect, through the effect on Ξ_{t-1} of $X_{t|t-j}$ on for $j \geq 1$ (due to (3.3)), which in turn affects Z_t through its effect upon i_t and $x_{t|t}$ (due to (3.1) and (3.2)).

In order to estimate X_t using a Kalman filter, we would like to find an indicator with the property that its innovation is a linear function of the forecast error, $X_t - X_{t|t-1}$, plus noise. The contemporaneous effect on Z_t means that its innovation does not meet this condition, since

$$Z_t - Z_{t|t-1} = L(X_t - X_{t|t-1}) + M(X_{t|t} - X_{t|t-1}) + v_t,$$

which also includes the terms $M(X_{t|t} - X_{t|t-1})$ (we have used that $\Xi_{t-1} = \Xi_{t-1|t-1}$). Thus, the contemporaneous effect enters via $MX_{t|t}$. In order to eliminate these effects of the estimated state upon the indicators, we might consider the vector of "ideal" indicators $\bar{Z}_t$, defined by the condition

$$\bar{Z}_t \equiv Z_t - M X_{t|t} - \Lambda\, \Xi_{t-1}, \tag{4.1}$$

where the contemporaneous effect is subtracted (the redundant component $\Lambda\, \Xi_{t-1}$ is also subtracted to get a more parsimonious indicator). These ideal indicators then have the desired property that their innovation is a linear function of the forecast error of the predetermined variables plus noise,

$$\begin{aligned} \bar{Z}_t &= L X_t + v_t, \\ \bar{Z}_t - \bar{Z}_{t|t-1} &= L(X_t - X_{t|t-1}) + v_t. \end{aligned}$$

However, these ideal indicators do not provide an *operational* way of eliminating the contemporaneous influence. Indeed, (4.1) is only an *implicit* definition, in the sense that the estimates $X_{t|t}$ that depend on the observable variables still enters into the identity and is assumed to be known. The ideal indicators can nonetheless provide a useful representation of the filtering problem for computational purposes, as we illustrate in the next section.

To get a recursive updating equation that is operational, we instead need one that only has current observable variables and previous estimates on the right side. We can use the prediction equation (3.10) ((2.21) in the discretion case) and solve for $X_{t|t}$ to get

$$X_{t|t} = (I + KM)^{-1}[(I - KL)X_{t|t-1} - K\Lambda\,\Xi_{t-1} + KZ_t], \tag{4.2}$$

where the matrix $I + KM$ must be invertible. We can then use (3.11) and (3.3) (where $\Xi_{t-1} \equiv 0$ in the discretion case) to express the dynamic equation for $X_{t|t}$ in terms of $X_{t-1|t-1}$ and Ξ_{t-2},

$$\begin{aligned} X_{t|t} &= (I + KM)^{-1}\{(I - KL)[(H + J)X_{t-1|t-1} + \Psi\,\Xi_{t-2}] - K\Lambda\,(SX_{t-1|t-1} + \Sigma\Xi_{t-2}) + KZ_t] \\ &= (I + KM)^{-1}\{[(I - KL)(H + J) - K\Lambda\,S]X_{t-1|t-1} + [(I - KL)\Psi - K\Lambda\,\Sigma]\Xi_{t-2} + KZ_t\}. \end{aligned} \tag{4.3}$$

Solving the system consisting of this equation and (3.3) backwards, we can express $X_{t|t}$ as the weighted sum of current and past observable variables,

$$X_{t|t} = \sum_{\tau=0}^{\infty} Q_\tau KZ_{t-\tau}, \tag{4.4}$$

where the matrix Q_τ is $[(I + KM)^{-1}(I - KL)(H + J)]^\tau$ in the discretion case and the upper left submatrix of the matrix

$$\begin{bmatrix} (I + KM)^{-1}[(I - KL)(H + J) - K\Lambda\,S] & (I + KM)^{-1}[(I - KL)\Psi - K\Lambda\,\Sigma] \\ S & \Sigma \end{bmatrix}^\tau$$

in the commitment case. The consequence of the contemporaneous effect via the matrix M only shows up in the premultiplication of the matrix $(I + KM)^{-1}$ above.

Thus, the evolution over time of the central bank's estimate of the predetermined states, and of the Lagrange multipliers needed to determine its action under the commitment equilibrium, can be expressed as a function of the observable variables. Furthermore, the Kalman gain matrix K gives the optimal weights on the vector of observable variables.. Row j of K gives the optimal weights in updating of element j of X_t. Column l of K gives the weights a particular observable variable Z_{lt} receives in updating the elements of X_t.

Since the estimate is a distributed lag of the observable variables, the estimate is updated only gradually. Thus, even under discretion, the observed policy will display considerable inertia, the more the noisier the current observables and the less the weight on current observations relative to previous estimates.

The elements of the Kalman gain matrix K depend upon the information structure (by (2.24) and (2.25) they depend on L, which by (2.19) depends on D^1, and on the covariance matrix Σ_{vv}). They also depend on part of the dynamics of the predetermined variables (by (2.25), they depend on H, which by (2.17) and (2.13) depends only on A^1, and on the covariance matrix Σ_{uu}). However, the elements of K are independent of the central-bank's objective, described by the matrices C^1, C^2, C_i, W and the discount factor δ, or, alternatively, of the central bank's reaction function (F, Φ) in (3.1) (where $\Phi = 0$ in the discretion case). This again illustrates the separation of the estimation problem from the optimization problem that arises under certainty-equivalence.

Suppose that, in row j of L, only one element is nonzero, say element (j, j). Then

$$Z_{jt} = X_{jt} + M_{j\cdot}X_{t|t} + \Lambda_{j\cdot}\Xi_{t-1} + v_{jt}$$

corresponds to an observation of X_{jt} with measurement error v_{jt} (we let $_{j\cdot}$ denote row j of a matrix, and we assume that element (j, j) of M, m_{jj}, fulfills $m_{jj} \neq -1$; this is now a necessary condition for the matrix $I+KM$ to be invertible). Suppose the variance of the measurement error approaches zero. Then the elements of row j in the Kalman gain matrix will approach zero, except the element (j, j) which approaches unity. This corresponds to X_{jt} being fully observable, resulting in

$X_{jt|t} = X_{jt}$. Suppose instead the variance of v_{jt} becomes unboundedly large. Then Z_{jt} is a useless indicator, and the Kalman gain matrix will assign a zero weight to this indicator; that is, all the elements in column j of K will be zero.

5 Example: Optimal monetary policy with unobservable potential output

Consider the following simple model, a variant of the model used, for example, in Clarida, Galí and Gertler [4], Woodford [41] and [42] and Svensson and Woodford [35]. The model equations are

$$\pi_t = \delta\pi_{t+1|t} + \kappa(y_t - \bar{y}_t) + \nu_t, \tag{5.1}$$

$$y_t = y_{t+1|t} - \sigma(i_t - \pi_{t+1|t}), \tag{5.2}$$

$$\bar{y}_{t+1} = \gamma\bar{y}_t + \eta_{t+1}, \tag{5.3}$$

$$\nu_{t+1} = \rho\nu_t + \varepsilon_{t+1}, \tag{5.4}$$

where π_t is inflation, y_t is (log) output, $\bar{y}_t$ is (log) potential output (the natural rate of output), ν_t is a serially correlated "cost-push" shock, and i_t is a one-period nominal interest rate (the central bank's monetary-policy instrument). In our specification of the exogenous disturbance processes, the shocks η_t and ε_t are iid with means zero and variances σ_η^2 and σ_ε^2, and the autoregressive coefficients γ and ρ satisfy $0 \leq \gamma, \rho < 1$. In our structural equations, the coefficient $0 < \delta < 1$ is also the discount factor for the central bank's loss function, and the coefficients κ and σ are positive.[8]

We assume a period loss function of the kind associated with flexible inflation targeting with a zero inflation target,[9]

$$L_t = \frac{1}{2}[\pi_t^2 + \lambda(y_t - \bar{y}_t)^2]. \tag{5.5}$$

[8] Note that $y_t - \bar{y}_t$ and ν_t here corresponds to x_t and u_t, respectively, in Svensson and Woodford [35]. Furthermore, current inflation and output are here forward-looking variables, whereas they are predetermined one period in [35]. The assumption that inflation and output are predetermined is arguably more realistic, but in the present context would not allow us to present a simple example in which one of the observables is a forward-looking variable. A more elaborate example (for instance, along the lines of Svensson [34]), that would be more realistic but less transparent in its analysis, would allow inflation and output to be predetermined, but introduce other forward-looking indicator variables, such as the exchange rate, a long bond rate, or other asset prices.

[9] See Woodford [40] for a welfare-theoretic justification of this loss function, in the case of exactly the microeconomic foundations that justify structural equations (5.1)–(5.2).

We assume that there is an imperfect observation, $\tilde{y}_t$, of potential output,

$$\tilde{y}_t = \bar{y}_t + \theta_t, \tag{5.6}$$

where the measurement error θ_t is iid with zero mean and variance σ_θ^2. We also assume that inflation is directly observable. Then the vector of observables is

$$Z_t = \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix}. \tag{5.7}$$

Since we assume that there are no unobservable shocks in the aggregate-demand equation, (5.2), in equilibrium output will be perfectly controllable. Then, we can consider a simplified variant of your model, with output as the control variable and consisting of the equations (5.1), (5.3) and (5.4). For the resulting equilibrium stochastic processes for y_t, $y_{t+1|t}$ and $\pi_{t+1|t}$, we can then use the aggregate-demand equation to infer the corresponding interest rates according to

$$i_t = \pi_{t+1|t} + \frac{1}{\sigma}(y_{t+1|t} - y_t). \tag{5.8}$$

We can now rewrite the model (5.1), (5.3) and (5.4) in the form (2.1),

$$\left[\begin{array}{c} X_{t+1} \\ \hline x_{t+1|t} \end{array}\right] \equiv \left[\begin{array}{c} \bar{y}_{t+1} \\ \nu_{t+1} \\ \hline \pi_{t+1|t} \end{array}\right] = \left[\begin{array}{cc|c} \gamma & 0 & 0 \\ 0 & \rho & 0 \\ \hline \kappa/\delta & -1/\delta & 1/\delta \end{array}\right] \left[\begin{array}{c} \bar{y}_t \\ \nu_t \\ \hline \pi_t \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \hline -\kappa/\delta \end{array}\right] y_t + \left[\begin{array}{c} \eta_{t+1} \\ \varepsilon_{t+1} \\ \hline 0 \end{array}\right], \tag{5.9}$$

where we let thin lines denote the decomposition of A^1 and B into its submatrices. We note that $E = 1$ and $A^2 = 0$. We can write the equation for the observables, (2.4), as

$$Z_t = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \bar{y}_t \\ \nu_t \\ \hline \pi_t \end{array}\right] + \left[\begin{array}{c} \theta_t \\ \hline 0 \end{array}\right],$$

which allows us to identify D^1, where the thin lines denotes its decomposition into D_1^1 and D_2^1, and v_t. We observe that $D^2 = 0$.

In this model, the central bank needs to form an estimate of the current potential output and cost-push shock, $\bar{y}_{t|t}$ and $\nu_{t|t}$, in order to set policy, the output level y_t. It observes an imperfect measure of potential output, $\tilde{y}_t$, and inflation, π_t, exactly. Since potential output is predetermined and independent both of current expectations and of the current instrument setting, noisy observation of it does not raise any special problems. In contrast, the observed inflation is here a forward-looking variable, which depends both on current expectations of future inflation and the current instrument setting. Current expectations and the instrument setting, furthermore, depend on the estimates of both current potential output and the current cost-push shock. These depend on the observation of inflation, completing the circle. Thus the central bank must sort through this simultaneity problem. Consequently our special case, in spite of its simplicity, incorporates the gist of the signal-extraction problem with forward-looking variables.

5.1 Equilibrium under discretionary optimization and under an optimal commitment

Due to the certainty-equivalence, in order to find the optimal policy, we can directly apply the solution of the full-information version of this model in Clarida, Galí and Gertler [4] and Svensson and Woodford [35]. Under discretionary optimization, the solution is[10]

$$\begin{aligned} y_t &= \bar{y}_{t|t} - \frac{\kappa}{\kappa^2 + \lambda(1-\delta\rho)}\nu_{t|t}, \\ \pi_t &= \frac{\lambda}{\kappa^2 + \lambda(1-\delta\rho)}\nu_{t|t} \end{aligned}$$

(where $\pi_t = \pi_{t|t}$ since inflation by assumption is directly observable). Under an optimal commitment, the solution is[11]

$$y_t = \bar{y}_{t|t} - \frac{\kappa}{\lambda}\frac{\mu}{1-\delta\rho\mu}\nu_{t|t} - \frac{\kappa}{\lambda}\mu\,\Xi_{t-1}, \tag{5.10}$$

$$\pi_{t|t} = \frac{\mu}{1-\delta\rho\mu}\nu_{t|t} - (1-\mu)\,\Xi_{t-1}, \tag{5.11}$$

$$\Xi_t = \frac{\mu}{1-\delta\rho\mu}\nu_{t|t} + \mu\,\Xi_{t-1}. \tag{5.12}$$

[10] See section 3.2 of Svensson and Woodford [35]. Recall that $y_t - \bar{y}_t$ and ν_t here corresponds to x_t and u_t, respectively, in [35]. Since the present model has an output target equal to potential output in the period loss function, (5.5), it corresponds to the case $x^* = 0$ in [35].

[11] See section 2.1 of Svensson and Woodford [35]. Note that Ξ_{t-1} here corresponds to φ_{t-1} in [35]. Because the present model corresponds to the case $x^* = 0$ in [35], $\varphi^* = 0$.

In the commitment case, Ξ_t is the Lagrange multiplier of the constraint corresponding to (5.1), the last row of (5.9), and μ ($0 < \mu < 1$) is a root of the characteristic equation of the difference equation for Ξ_t that results from substitution of the first-order conditions into (5.1).

5.2 An optimal targeting rule

The above characterization of the optimal commitment allows us to derive a simple targeting rule, that represents one practical approach to the implementation of optimal policy, as discussed in Svensson and Woodford [35]. By (5.10) and (5.12), we have

$$y_t - \bar{y}_{t|t} = -\frac{\kappa}{\lambda}\,\Xi_t, \tag{5.13}$$

and by (5.11) and (5.12), we have

$$\pi_t = \Xi_t - \Xi_{t-1}. \tag{5.14}$$

These are just the first-order conditions under commitment, the combination of which with the dynamic equations (5.1), (5.3) and (5.4) then result in (5.10)–(5.12). We can furthermore eliminate the Lagrange multipliers from (5.13) and (5.14) and get a consolidated first-order condition

$$\pi_t = -\frac{\lambda}{\kappa}[(y_t - \bar{y}_{t|t}) - (y_{t-1} - \bar{y}_{t-1|t-1})]. \tag{5.15}$$

In the full-information case, $\bar{y}_t$ and $\bar{y}_{t-1}$ would be substituted for $\bar{y}_{t|t}$ and $\bar{y}_{t-1|t-1}$ in (5.15). As discussed in detail in [35], the full-information analogue of (5.15) can be interpreted as a *targeting rule*, which if followed by the central bank will result in the full social optimum under commitment (when the intertemporal loss function with the period loss function (5.5) is interpreted as the social loss function). Thus, inflation should be adjusted to equal the negative change in the output gap, multiplied by the factor λ/κ.

This targeting rule is remarkable in that it only depends on the relative weight on output-gap stabilization in the loss function, λ, and the slope of the short-run Phillips curve, κ. In particular, the targeting rule is robust to the number and stochastic properties of additive shocks to the aggregate-supply equation (as witnessed by the lack of dependence on the AR(1) coefficient of the cost-push shock, ρ, and the variances of the iid shock, σ_ε^2) and (as long as the interest rate does not enter the loss function) completely independent of the aggregate-demand equation (5.2).

An alternative formulation of the targeting rule is in terms of a target for the price *level*, rather than the inflation rate. We observe that (5.15) implies that

$$p_t - p^* = -\frac{\lambda}{\kappa}(y_t - \bar{y}_{t|t}), \tag{5.16}$$

where p_t is the (log) price level ($\pi_t \equiv p_t - p_{t-1}$) and p^* is a constant that can be interpreted as an implicit price-level target. Similarly, (5.16) implies (5.15), so these are equivalent targeting rules, each equally consistent with the optimal commitment. (It is worth noting that under our informational assumptions, p_t is also public information at date t.) This illustrates the close relation between inflation targeting under commitment and price-level targeting, further discussed in Vestin [38], Svensson [31] and [33] and Woodford [41] and [42]. We also note that under the optimal commitment, the Lagrange multipliers satisfy

$$\Xi_t = p_t - p^*.$$

This is useful below as an empirical proxy for variation in the Lagrange multipliers.

An interesting feature of both of these characterizations of optimal policy is that, under partial information, the targeting rule has *exactly* the same form as under full information, except that the *estimated* output gap, $y_t - y_{t|t}$, is consistently substituted for the actual output gap, $y_t - \bar{y}_t$). Thus, policy should respond to exactly the same extent to the estimated output gap under partial information as to the actual output gap under full information. This is an important illustration of the certainty-equivalence result demonstrated earlier in the paper.

However, it is important to note that the targeting rules (5.15) and (5.16) are written in terms of the *optimal estimate* of the output gap, $y_t - \bar{y}_{t|t}$, *not* in terms of the output gap measure $y_t - \tilde{y}_t$ implied by the imperfect observation of potential output, $\tilde{y}_t$. As we shall see, the optimal degree of response to an imperfect observation of the output gap does indeed depend on the degree of noise in the observation.

5.3 Ideal indicators and optimal filtering

Let us return to the solutions under discretion and commitment. It follows that we can write these as

$$y_t = \begin{bmatrix} 1 & f \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Phi\,(p_{t-1} - p^*), \tag{5.17}$$

$$\pi_t = \begin{bmatrix} 0 & g \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Gamma\,(p_{t-1} - p^*), \tag{5.18}$$

$$\Xi_t = \begin{bmatrix} 0 & g \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Sigma\,(p_{t-1} - p^*), \tag{5.19}$$

where $p_t = \pi_t + p_{t-1}$. Under discretion, we have

$$f = -\frac{\kappa}{\kappa^2 + \lambda(1 - \delta\rho)}, \quad g = -\frac{\lambda}{\kappa} f = \frac{\lambda}{\kappa^2 + \lambda(1 - \delta\rho)}, \quad \Phi = \Gamma = \Sigma = 0.$$

This allows us to identify the matrices F and G in (2.7) and (2.8). Under commitment, we have

$$f \equiv -\,(\kappa/\lambda)[\mu/(1{-}\delta\rho\mu)], \;\; \Phi \equiv -(\kappa/\lambda)\mu, \;\; g = -\frac{\lambda}{\kappa} f = \mu/(1{-}\delta\rho\mu), \;\; \Gamma = -(1{-}\mu), \;\; \Sigma = \mu.$$

This allows us to identify the matrices F, Φ, G and Γ in (3.1)–(3.3).

Furthermore, π_t will be given by

$$\pi_t = \begin{bmatrix} -\kappa & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \kappa & g-1 \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Gamma\,(p_{t-1} - p^*), \tag{5.20}$$

where $\Gamma \equiv 0$ under discretion. The last equation allows the identification of the matrices G^1 and G^2 in (2.12) and (3.5). We are then able to compute the matrices

$$H = \begin{bmatrix} \gamma & 0 \\ 0 & \rho \end{bmatrix}, \quad J = 0, \quad L = \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ \kappa & g-1 \end{bmatrix}. \tag{5.21}$$

Furthermore, the matrices Ψ and Λ in the commitment case are given by

$$\Psi = 0, \quad \Lambda = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}.$$

In order to solve the estimation problem in this special case, we need to find the 2×2 Kalman gain matrix, K, given by (2.24), where the 2×2 matrix of forecast errors, P, is given by (2.25). The updating equation (3.10) can then be written

$$\begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} = \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} + K\left(\begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix} - L \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} - M \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} - \Lambda\left(p_{t-1} - p^*\right)\right), \tag{5.22}$$

where $\Lambda \equiv 0$ under discretion. This can be written more simply as

$$\begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} = \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} + K\left[\bar{Z}_t - \bar{Z}_{t|t-1}\right]$$

in terms of the *ideal indicators* $\bar{Z}_t$ given by

$$\bar{Z}_t \equiv \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix} - M \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} - \Lambda\left(p_{t-1} - p^*\right) = \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t - \kappa \bar{y}_{t|t} - (g-1)\upsilon_{t|t} - \Gamma\left(p_{t-1} - p^*\right) \end{bmatrix}. \tag{5.23}$$

Combining (5.20) and the second row in (5.23), we see that the ideal indicators in fact correspond to

$$\bar{Z}_t = \begin{bmatrix} \bar{y}_t + \theta_t \\ -\kappa \bar{y}_t + \nu_t \end{bmatrix}. \tag{5.24}$$

Thus, the filtering problem may be reduced to one of observing a noisy measure of potential output along with a linear combination of potential output and the cost-push shock. That observation of the forward-looking inflation rate implies the observability of this linear combination of the potential output and cost-push shock is quite intuitive. From the aggregate supply equation (5.1) we see that in equilibrium observability of π_t, $\pi_{t+1|t}$ and y_t implies that the remainder $-\kappa\bar{y}_t + \nu_t$ must be observable as well.

The ideal indicators are not operational, as their construction presumes that $\bar{y}_{t|t}$ and $\nu_{t|t}$ are already known. However, consideration of the simple problem that would result if these indicators were available is useful, as a way of determining the Kalman gain matrix K. This estimation problem consists of the simple transition equation

$$\begin{bmatrix} \bar{y}_{t+1} \\ \nu_{t+1} \end{bmatrix} = H \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix}, \tag{5.25}$$

where H is given by (5.3) and (5.21), and the measurement equation (5.24). (The transition equation is this simple because the predetermined variables, $\bar{y}_t$ and ν_t, are exogenous, that is, $A_{12}^1 = 0$, $A_{11}^2 = 0$, $A_{12}^2 = 0$, $B_1 = 0$.) In appendix C, we derive an analytical expression for the Kalman gain matrix and show that it is of the form

$$K = \begin{bmatrix} k_{11} & k_{12} \\ \kappa k_{11} & \kappa k_{12} + 1 \end{bmatrix}. \tag{5.26}$$

Here

$$k_{11} \equiv \frac{q}{\sigma_\theta^2} \tag{5.27}$$

and q is the positive root of a quadratic equation, which depends on κ, γ, ρ and the variances σ_η^2, σ_ε^2 and σ_θ^2. The element k_{12} is also reported as a function of these parameters in the appendix.

Having determined K, we may return to the consideration of an operational procedure for computing the optimal estimates of the underlying exogenous disturbances. For this the central bank can use the operational recursive updating equations (4.2) and (4.3), which can be written

$$\begin{aligned} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} &= (I + KM)^{-1} \left((I - KL) \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} - K\Lambda(p_{t-1} - p^*) + KZ_t \right) \\ &= (I + KM)^{-1} \left((I - KL)H \begin{bmatrix} \bar{y}_{t-1|t-1} \\ \nu_{t-1|t-1} \end{bmatrix} - K\Lambda(p_{t-1} - p^*) + KZ_t \right) \end{aligned} \tag{5.28}$$

This last equation is simpler than (4.3) because in this example, $J = 0$ and $\Psi = 0$.

Equation (5.28) allows us to solve for the optimal estimates $\bar{y}_{t|t}$ and $\nu_{t|t}$ as functions of the history of observables ($\tilde{y}_\tau$ and π_τ or, equivalently, $\tilde{y}_\tau$ and p_τ) up through period t. This solution for $\bar{y}_{t|t}$ can then be substituted into (5.16), to obtain an equation for y_t as a function of the history of $\tilde{y}_\tau$ and p_τ. If y_t

were actually the central bank's instrument, this would then represent a rule for setting that instrument as a function of the observables. However, in practice, a central bank has no direct control over current output, and instead typically uses a short-term nominal interest rate as its instrument. Derivation of an instrument rule then requires that we consider the evolution of nominal interest rates implied by the above characterization of the optimal commitment.

5.4 An optimal instrument rule

We consider the evolution of the interest rate i_t under the optimal commitment. The solution for output and inflation are given by (5.17) and (5.18). Combining these with (5.8) results, after simplification, in the instrument rule

$$i_t = \tilde{F} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \tilde{\Phi}(p_{t-1} - p^*) \tag{5.29}$$

in terms of responses to the current estimates of the predetermined variables and the lagged price level, where

$$\begin{aligned} \tilde{F} &= GH + FS + \frac{1}{\sigma}[F(H - I) + \Phi S] \\ \tilde{\Phi} &= \Gamma\Sigma + \frac{1}{\sigma}\Phi(\Sigma - I). \end{aligned}$$

(Note that discretionary optimization corresponds to a similar instrument rule, in which however $\tilde{\Phi} = 0$.) Certainty equivalence implies that the matrices $\tilde{F}$ and $\tilde{\Phi}$ are independent of the variances of the shocks, σ_η^2, σ_ε^2 and σ_θ^2.

As in the previous subsection, we can utilize (5.28) to express the instrument rule in terms of current observables, lagged estimates and the lagged price level. Let us focus on the response of the interest rate to the current observables, for given levels of lagged estimates and price level. This response is by (4.3) and (5.29) given by

$$\tilde{F}(I + KM)^{-1}(K_1\tilde{y}_t + K_2\pi_t),$$

where we have partitioned the Kalman gain matrix according to $K = [K_1 \;\; K_2]$, so K_1 is the first column in (5.26).

Of course, this response to the observables, via the Kalman gain matrix, depends on the variances of the shocks. In particular, we can examine how the response to the observation of the potential output, $\tilde{y}_t$, depends on its noise, i.e., the variance of its measurement error σ_θ^2. In appendix C, we show that the root q in (5.27) remains positive and bounded for all positive σ_θ^2. This means that k_{11} approaches zero when degree of noise becomes large. Thus, the optimal weights on the observation of potential output in the submatrix K_1 goes to zero when its information content goes to zero. This is an example of the Kalman filter assigning zero weight to useless indicators, mentioned in section 4.

Again, this does not mean that the response to the optimal estimate of potential output, $\bar{y}_{t|t}$, changes. By certainty-equivalence, it stays the same. It is only that the *direct observation* of potential output, $\tilde{y}_t$, is disregarded in the construction of the optimal estimate. Instead, in this case the central bank will rely only on the observed inflation rate.

6 Conclusions

In this paper, we have restated the important result that, under symmetric partial information, certainty-equivalence and the separation principle continue to hold in the case of linear rational-expectations models and a quadratic loss function. Then optimal policy as a function of the current estimate of the state of the economy is the same as if the state were observed.

However, policy as a function of the observable variables (and the actual, as distinct from the estimated, state of the economy) will display considerable inertia, since the current estimate will be a distributed lag of the current and past observable variables (and actual states of the economy). Thus,discretionary policy—which as discussed in Woodford [41] and [42] and Svensson and Woodford [35], often lacks the history-dependence that characterizes optimal policy under commitment—will in this case display a certain inertial character as a consequence of partial information. It seems likely that this inertial character will be more pronounced the noisier the information in the observable variables, as this should lead to slower updating of the current estimate of the state of the economy. To what extent this may affect the welfare comparison between discretionary policy and the optimal policy under commitment (which represents the social optimum), is a topic for future research.

Even given certainty-equivalence and the separation principle, the estimation problem with forward-looking observable variables presents a challenge, due to the circularity in the way that the observable variables both affect and depend on the current estimate. The optimal operational Kalman filter under these circumstances needs to be modified to circumvent that circularity, as we have shown.

Our results have been derived under the assumption of symmetric information between the central bank and the aggregate private sector, as a result of which certainty-equivalence and the separation between optimization and estimation hold. This case seems to us to be of practical interest, since we believe that any informational advantage of central banks consists mainly of better information about their own intentions (as in the papers of Cukierman and Meltzer [5] and Faust and Svensson [9]). Any such private information is nowadays increasingly being eroded by the general tendency toward increased transparency in monetary policy, whether willingly adopted by the central banks or, in some cases, forced upon them by irresistible outside demands. Nevertheless, it is of interest to understand how these results are modified when there is asymmetric information (especially in the direction of central banks having less information than parts of the private sector); this topic is taken up in Svensson and Woodford [36].

We have illustrated our general results in terms of a forward-looking model of monetary policy with unobservable potential output and a partially observable cost-push shock, where the observable variables both affect and depend on the current estimates of potential output and the cost-push shock. This situation is obviously highly relevant for many central banks, including the recently established Eurosystem. We note that our analysis of optimal policy does imply an important role for an estimate of current potential output, and that the proper weight to be put on such an *estimate* under an optimal policy rule is unaffected by the degree of noise in available measures of potential output. Thus the lack of more accurate measures is not a reason for policy to respond less to perceived fluctuations in the output gap (though inaccuracy of particular indicators can be a reason for a bank's *estimate* of potential output to be less influenced by those indicators).

On the other hand, in the case of pure indicator variables—variables that are neither target variables (variables that enter the loss function) nor direct causal determinants of target variables, and that accordingly would not be responded to under an optimal policy in the case of full information—the degree to which monetary policy should take account of them is definitely dependent upon how closely they are in fact associated with the (causal) state variables that one seeks to estimate. This precept does not always play as large a role in current central banking practice as it might.

As an example, the Eurosystem has put special emphasis on one particular indicator, the growth of Euro-area M3 relative to a reference value of 4.5 percent per year, elevating this money-growth indicator to the status of one of two "pillars" of the Eurosystem monetary strategy (in addition to "a broadly-based assessment of the outlook for future price developments").[12] Money growth in excess of the reference value is supposed to indicate "risks to price stability." As discussed by commentators such as Svensson [30], Rudebusch and Svensson [21] and Gerlach and Svensson [11], it is difficult to find rational support for this prominence of the money-growth indicator. Instead, monetary aggregates would seem to be properly viewed as just one set of indicators among many others, the relative weight on which should exclusively depend on their performance in predicting the relevant aspects of the current state of the economy; more specifically, how useful current money growth is as an input in conditional forecasts of inflation some two years ahead.

Under normal circumstances, the information content of money growth for inflation forecasts in the short and medium term seems to be quite low.[13] Only

[12] See, for instance, European Central Bank [8].

[13] See Estrella and Mishkin [7] and Stock and Watson [26]; Gerlach and Svensson [11] find, for reconstructed Euro-area data, information for future inflation in another monetary indicator,

in the long run does a high correlation between money growth a inflation result. Under the special circumstances of the introduction of a new common currency, the demand for money is likely to be quite unpredictable and possibly very unstable, since important structural changes are likely to occur in financial markets and banking. Under such circumstances, the information content of money is likely on theoretical grounds to be even lower than under normal circumstances. Thus the uncertainty associated with the introduction of the new currency should provide an argument for relying less, rather than more, on monetary aggregates as indicators.

---

the "real money gap," but little or no information in the Eurosystem's money-growth indicator.

A Optimization under discretion and certainty-equivalence

Consider the decision problem to choose i_t in period t to minimize (2.6) (with $0 < \delta < 1$) under discretion, that is, subject to (2.1)–(2.5) and

$$i_{t+1} = F_{t+1} X_{t+1|t+1} \tag{A.1}$$

$$x_{t+1|t+1} = G_{t+1} X_{t+1|t+1}, \tag{A.2}$$

where F_{t+1} and G_{t+1} are determined by the decision problem in period $t+1$.

For the full information case, Oudiz and Sachs [17] have derived an algorithm for the discretionary equilibrium, which is further discussed in Backus and Driffill [2] and Currie and Levin [6].[14] Following Pearlman [19], but with a more explicit proof, this appendix shows that this algorithm, appropriately adapted, is valid also for the partial-information case.

First, using (A.2), taking expectations in period t of the upper block of (2.1), and using (2.10), we get

$$x_{t+1|t} = G_{t+1} X_{t+1|t} = G_{t+1}(A_{11} X_{t|t} + A_{12} x_{t|t} + B_1 i_t). \tag{A.3}$$

Taking the expectation in period t of the lower block of (2.1), we get

$$E x_{t+1|t} = A_{21} X_{t|t} + A_{22} x_{t|t} + B_2 i_t \tag{A.4}$$

(recall that E is a matrix and not the expectations operator). Multiplying (A.3) by E, setting the result equal to (A.4) and solving for $x_{t|t}$ gives

$$x_{t|t} = \tilde{A}_t X_{t|t} + \tilde{B}_t i_t, \tag{A.5}$$

where

$$\tilde{A}_t \equiv (A_{22} - E G_{t+1} A_{12})^{-1} (E G_{t+1} A_{11} - A_{21}),$$

$$\tilde{B}_t \equiv (A_{22} - E G_{t+1} A_{12})^{-1} (E G_{t+1} B_1 - B_2)$$

(we assume that $A_{22} - E G_{t+1} A_{12}$ is invertible). Using (A.5) in the expectation of the upper block of (2.1) then gives

$$X_{t+1|t} = A_t^* X_{t|t} + B_t^* i_t, \tag{A.6}$$

[14] See Söderlind [25] for a detailed presentation.

where

$$\begin{aligned} A_t^* &\equiv A_{11} + A_{12}\tilde{A}_t, \\ B_t^* &\equiv B_1 + A_{12}\tilde{B}_t. \end{aligned}$$

Second, by (2.2) and (2.3) we can write

$$L_{t|t} = \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix}' Q \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + 2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix}' U i_t + i_t' R i_t + l_t, \tag{A.7}$$

where

$$C \equiv C^1 + C^2, \;\; Q \equiv C'WC, \;\; U \equiv C'WC_i, \;\; R \equiv C_i'WC_i$$

$$l_t \equiv \mathrm{E}\left\{ \begin{bmatrix} X_t - X_{t|t} \\ x_t - x_{t|t} \end{bmatrix}' C^{1\prime} W C^1 \begin{bmatrix} X_t - X_{t|t} \\ x_t - x_{t|t} \end{bmatrix} |I_t \right\}. \tag{A.8}$$

Using (A.5) in (A.7) leads to

$$L_{t|t} = X_{t|t}' Q_t^* X_{t|t} + 2X_{t|t}' U_t^* i_t + i_t' R_t^* i_t + l_t, \tag{A.9}$$

where

$$\begin{aligned} Q_t^* &\equiv Q_{11} + Q_{12}\tilde{A}_t + \tilde{A}_t' Q_{21} + \tilde{A}_t' Q_{22} \tilde{A}_t, \\ U_t^* &\equiv Q_{12}\tilde{B}_t + \tilde{A}_t' Q_{22} \tilde{B}_t + U_1 + \tilde{A}_t' U_2, \\ R_t^* &\equiv R + \tilde{B}_t' Q_{22} \tilde{B}_t + \tilde{B}_t' U_2 + U_2' \tilde{B}_t, \end{aligned}$$

and Q and U are decomposed according to $X_{t|t}$ and $x_{t|t}$.

Third, since the loss function is quadratic and the constraints are linear, it follows that the optimal value of the problem will be quadratic. In period $t+1$ the optimal value will depend on the estimate $X_{t+1|t+1}$ and can hence be written $X_{t+1|t+1}' V_{t+1} X_{t+1|t+1} + w_{t+1}$, where V_{t+1} is a positive semidefinite matrix and w_{t+1} is a scalar. Then the optimal value of the problem in period t is associated with the positive semidefinite matrix V_t and the scalar w_t, and fulfills the Bellman equation

$$X_{t|t}' V_t X_{t|t} + w_t \equiv \min_{i_t} \left\{ L_{t|t} + \delta \mathrm{E}[X_{t+1|t+1}' V_{t+1} X_{t+1|t+1} + w_{t+1} | I_t] \right\}, \tag{A.10}$$

subject to (A.6) and (A.9). Indeed, the problem has been transformed to a standard linear regulator problem without forward-looking variables, albeit in terms of $X_{t|t}$ and with time-varying parameters. The first-order condition is, by (A.9) and (A.10),

$$\begin{aligned} 0 &= X'_{t|t}U_t^* + i'_t R_t^* + \delta \mathrm{E}[X'_{t+1|t+1}V_{t+1}B_t^*|I_t] \\ &= X'_{t|t}U_t^* + i'_t R_t^* + \delta(X'_{t|t}A_t^{*\prime} + i'_t B_t^{*\prime})V_{t+1}B_t^*. \end{aligned}$$

Here we have assumed that l_t is independent of i_t, which assumption is verified below. The first-order condition can be solved for the reaction function

$$i_t = F_t X_{t|t}, \tag{A.11}$$

where

$$F_t \equiv -\left(R_t^* + \delta B_t^{*\prime} V_{t+1} B_t^*\right)^{-1}\left(U_t^{*\prime} + \delta B_t^{*\prime} V_{t+1} A_t^*\right)$$

(we assume that $R_t^* + \delta B_t^{*\prime} V_{t+1} B_t^*$ is invertible). Using (A.11) in (A.5) gives

$$i_t = G_t X_{t|t},$$

where

$$G_t \equiv \tilde{A}_t + \tilde{B}_t F_t.$$

Furthermore, using (A.11) in (A.10) and identifying gives

$$V_t \equiv Q_t^* + U_t^* F_t + F_t' U_t^{*\prime} + F_t' R_t^* F_t + \delta \left(A_t^* + B_t^* F_t\right)' V_{t+1} \left(A_t^* + B_t^* F_t\right).$$

Finally, the above equations define a mapping from $(F_{t+1}, G_{t+1}, V_{t+1})$ to (F_t, G_t, V_t). The solution to the problem is a fixpoint (F, G, V) of the mapping. It is obtained as the limit of (F_t, G_t, V_t) when $t \to -\infty$. The solution thus fulfills the corresponding steady-state matrix equations. Thus, the instrument i_t and the estimate of the forward-looking variables $x_{t|t}$ will be linear functions, (2.7) and (2.8) of the estimate of the predetermined variables $X_{t|t}$, where the corresponding F and G fulfill the corresponding steady-state equations. In particular, G will fulfill (2.9).

It also follows that F, G and V only depend on $A \equiv A^1+A^2$, B, $C \equiv C^1+C^2$, C_i, E, W and δ and are independent of D^1, D^2, Σ_{uu} and Σ_{vv}. This demonstrates the certainty-equivalence of the discretionary equilibrium.

It remains to verify the assumption that l_t in (A.8) is independent of i_t. Since by (2.12)–(2.13), $x_t - x_{t|t} = -(A^1_{22})^{-1}A^1_{21}(X_t - X_{t|t})$, it is sufficient to demonstrate that $\mathrm{E}[(X_t - X_{t|t})(X_t - X_{t|t})'|I_t]$ is independent of i_t. By (2.22),

$$X_t - X_{t|t} = X_t - X_{t|t-1} + K(L(X_t - X_{t|t-1}) + v_t = (I + KL)(X_t - X_{t|t-1}) + Kv_t.$$

Since X_t and $X_{t|t-1}$ are predetermined and v_t is exogenous, the assumption is true.

B The Kalman gain matrix and the covariance of the forecast errors

It is practical to express the dynamics in terms of the prediction errors of X_t and Z_t, relative to period $t-1$ information,

$$\begin{aligned} \tilde{X}_t &\equiv X_t - X_{t|t-1}, \\ \tilde{Z}_t &\equiv Z_t - Z_{t|t-1} = Z_t - (L+M)X_{t|t-1}, \end{aligned}$$

where we have used (2.16). Then the prediction equation can be written

$$X_{t|t} = X_{t|t-1} + K(L\tilde{X}_t + v_t). \tag{B.1}$$

First, (2.16) implies that

$$Z_{t|t-1} = (L+M)X_{t|t-1}$$

and hence that

$$\tilde{Z}_t = L\tilde{X}_t + M(X_{t|t} - X_{t|t-1}) + v_t$$

Substitution of (B.1) into this then yields

$$\tilde{Z}_t = (I + MK)(L\tilde{X}_t + v_t). \tag{B.2}$$

Thus we get the desired expression

$$\tilde{Z}_t = N\tilde{X}_t + \nu_t, \tag{B.3}$$

where

$$N \equiv (I + MK)L, \tag{B.4}$$

$$\nu_t \equiv (I + MK)v_t. \tag{B.5}$$

In order to find the dynamics for the prediction error $\tilde{X}_t$, we subtract (2.23) from (2.15) and use (B.1), which gives

$$\tilde{X}_{t+1} = H(X_t - X_{t|t}) + u_{t+1} = H\tilde{X}_t - HK(L\tilde{X}_t + v_t) + u_{t+1}.$$

Hence we get the desired expression

$$\tilde{X}_{t+1} = T\tilde{X}_t + \omega_{t+1}, \tag{B.6}$$

where

$$T \equiv H(I - KL), \tag{B.7}$$

$$\omega_{t+1} \equiv u_{t+1} - HKv_t. \tag{B.8}$$

Now, (B.6) and (B.3) can be seen as the transition and measurement equations, respectively, for a standard Kalman-filter problem for the unobservable variable $\tilde{X}_t$ with $\tilde{Z}_t$ being the observable variable. Consequently, the prediction equation for $\tilde{X}_{t|t}$ can be written

$$\tilde{X}_{t|t} = PN'(NPN' + \Sigma_{\nu\nu})^{-1}(N\tilde{X}_t + \nu_t) \tag{B.9}$$

where $'$ denotes transpose and where we have used $\tilde{X}_{t|t-1} \equiv 0$ and $P \equiv \mathrm{Cov}[\tilde{X}_t - \tilde{X}_{t|t-1}] = \mathrm{Cov}[\tilde{X}_t]$ is the covariance matrix for the prediction errors (see appendix D). By (B.6) we directly get

$$P = TPT' + \Sigma_{\omega\omega}. \tag{B.10}$$

We also have

$$\Sigma_{\nu\nu} = \mathrm{E}[\nu_t \nu_t'] = (I + MK)\Sigma_{vv}(I + MK)', \tag{B.11}$$

$$\Sigma_{\omega\omega} = HK\Sigma_{vv}K'H' + \Sigma_{uu}. \tag{B.12}$$

We express $X_{t|t}$ in terms of the prediction error $\tilde{Z}_t$ by solving for $X_{t|t}$ in (2.21), which gives

$$\begin{aligned}
X_{t|t} &= (I+KM)^{-1}[X_{t|t-1} + K(Z_t - LX_{t|t-1})] \\
&= X_{t|t-1} + (I+KM)^{-1}K[Z_t - (L+M)X_{t|t-1}] \\
&= X_{t|t-1} + (I+KM)^{-1}K\tilde{Z}_t \\
&= X_{t|t-1} + K(I+MK)^{-1}\tilde{Z}_t, \qquad \text{(B.13)}
\end{aligned}$$

where we have used the convenient identities $(I+KM)^{-1} \equiv I-(I+KM)^{-1}KM$ and $(I+KM)^{-1}K \equiv K(I+MK)^{-1}$.

Now, comparing (B.9) and (B.13), using (B.3) and $\tilde{X}_{t|t} = X_{t|t} - X_{t|t-1}$, we see that

$$K(I+MK)^{-1} = PN'(NPN' + \Sigma_{\nu\nu})^{-1}.$$

Substituting (B.4) for N and (B.11) for in the right side, we get the final expression for K, (2.24).

Substituting (2.24) for K in T in (B.7) and (B.10) then gives the final equation for P, (2.25).

C The Kalman gain matrix in the example of section 5

The transition equation and measurement equations are given by

$$\begin{bmatrix} \bar{y}_{t+1} \\ \nu_{t+1} \end{bmatrix} = H \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix},$$

$$\bar{Z}_t = L \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + v_t,$$

where H and L are given by (5.21) and $v_t \equiv \begin{bmatrix} \theta_t \\ 0 \end{bmatrix}$. Since L is invertible in this case, it is practical to do a variable transformation of the predetermined variables such that the corresponding L-matrix in the measurement equation is the identity matrix. Thus,

$$\bar{X}_t \equiv \begin{bmatrix} \bar{y}_t \\ -\kappa\bar{y}_t + \nu_t \end{bmatrix} = L \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix},$$

in which case the transition and measurement equations are

$$
\begin{aligned}
\bar{X}_{t+1} &= \bar{H}\bar{X}_t + \bar{u}_{t+1}, \\
\bar{Z}_t &= \bar{X}_t + v_t,
\end{aligned}
$$

where

$$
\bar{H} \equiv LHL^{-1} = \begin{bmatrix} \gamma & 0 \\ \kappa(\rho-\gamma) & \rho \end{bmatrix}, \quad \bar{u}_t \equiv L \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \eta_t \\ -\kappa\eta_t + \varepsilon_t \end{bmatrix},
$$

$$
\Sigma_{\bar{u}\bar{u}} = \begin{bmatrix} \sigma_\eta^2 & -\kappa\sigma_\eta^2 \\ -\kappa\sigma_\eta^2 & \kappa^2\sigma_\eta^2 + \sigma_\varepsilon^2 \end{bmatrix}, \quad \Sigma_{vv} = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & 0 \end{bmatrix}.
$$

In order to determine the Kalman gain matrix for the transformed variables, we need to know the covariance matrix of the corresponding one-period-ahead forecast errors, $\bar{P} \equiv \mathrm{Var}[\bar{X}_t - \bar{X}_{t|t-1}]$. First, we note that the current forecast-error covariance matrix Q fulfills

$$
Q \equiv \mathrm{Var}[\bar{X}_t - \bar{X}_{t|t}] = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix},
$$

where $q \equiv \mathrm{Var}[\bar{y}_t - \bar{y}_{t|t}]$ is the current forecast error for potential output and remains to be determined, and we have used that $-\kappa\bar{y}_t + \nu_t$ is observed without error. Then $\bar{P}$ depends on Q according to

$$
\bar{P} = \bar{H}Q\bar{H}' + \Sigma_{\bar{u}\bar{u}}. \tag{C.1}
$$

Furthermore, Q depends on $\bar{P}$ according to the updating equation

$$
Q = \bar{P} - \bar{P}(\bar{P} + \Sigma_{vv})^{-1}\bar{P}. \tag{C.2}
$$

We can rewrite this equation as

$$
Q(I + \bar{P}^{-1}\Sigma_{vv}) = \Sigma_{vv}.
$$

Then we can exploit that Q and Σ_{vv} are nonzero only in their (1,1) elements, so the matrix equation reduces to the single equation

$$
q\left(1 + \bar{P}^{-1}{}_{11}\sigma_\theta^2\right) = \sigma_\theta^2, \tag{C.3}
$$

where $\bar{P}^{-1}{}_{ij}$ denotes the (i,j) element of the inverse of $\bar{P}$ (*not* the inverse of the (i,j) element of $\bar{P}$).

In order to solve this equation for q, we need to express this element of the inverse in terms of q. Substitution of $\bar{H}$, Q and $\Sigma_{\bar{u}\bar{u}}$ in (C.1) results in

$$\begin{aligned}\bar{P} &= q\begin{bmatrix}\gamma^2 & \gamma\kappa(\rho-\gamma)\\ \gamma\kappa(\rho-\gamma) & \kappa^2(\rho-\gamma)^2\end{bmatrix}+\begin{bmatrix}\sigma_\eta^2 & -\kappa\sigma_\eta^2\\ -\kappa\sigma_\eta^2 & \kappa^2\sigma_\eta^2+\sigma_\varepsilon^2\end{bmatrix}\\ &= \begin{bmatrix}\gamma^2 q+\sigma_\eta^2 & \gamma\kappa(\rho-\gamma)q-\kappa\sigma_\eta^2\\ \gamma\kappa(\rho-\gamma)q-\kappa\sigma_\eta^2 & \kappa^2(\rho-\gamma)^2 q+\kappa^2\sigma_\eta^2+\sigma_\varepsilon^2\end{bmatrix}.\end{aligned}$$

We then have

$$\bar{P}^{-1}{}_{11} = \frac{\kappa^2(\rho-\gamma)^2 q+\kappa^2\sigma_\eta^2+\sigma_\varepsilon^2}{|\bar{P}|}, \tag{C.4}$$

$$\bar{P}^{-1}{}_{12} = -\frac{\gamma\kappa(\rho-\gamma)q-\kappa\sigma_\eta^2}{|\bar{P}|},$$

where

$$|\bar{P}| = [\gamma^2\sigma_\varepsilon^2+(\kappa\rho)^2\sigma_\eta^2)]q+\sigma_\eta^2\sigma_\varepsilon^2.$$

Using (C.4) in (C.3) results in the quadratic equation

$$aq^2+bq+c=0, \tag{C.5}$$

where

$$\begin{aligned}a &\equiv \kappa^2(\rho-\gamma)^2\sigma_\theta^2+(\kappa\rho)^2\sigma_\eta^2+\gamma^2\sigma_\varepsilon^2>0, &\text{(C.6)}\\ b &\equiv [\kappa^2(1-\rho^2)\sigma_\eta^2+(1-\gamma^2)\sigma_\varepsilon^2]\sigma_\theta^2+\sigma_\eta^2\sigma_\varepsilon^2>0, &\text{(C.7)}\\ c &\equiv -\sigma_\eta^2\sigma_\varepsilon^2\sigma_\theta^2<0. &\text{(C.8)}\end{aligned}$$

The signs of a, b and c imply that the quadratic equation has two real roots, one positive and one negative. The positive root is the only possible value for the forecast-error variance q, so we obtain

$$q=\frac{-b+\sqrt{b^2-4ac}}{2a}>0. \tag{C.9}$$

Having determined q, we can now express the Kalman gain matrix in terms of q. The Kalman gain matrix $\bar{K}$ for the estimation of the transformed variables $\bar{X}_t$ is given by

$$\bar{K}=\bar{P}(\bar{P}+\Sigma_{vv})^{-1}=I-Q\bar{P}^{-1},$$

where we have used (C.2). Using the form of Q, we then get

$$\bar{K} = \begin{bmatrix} 1 - q\bar{P}^{-1}{}_{11} & -q\bar{P}^{-1}{}_{12} \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} k_{11} & k_{12} \\ 0 & 1 \end{bmatrix}.$$

From (C.3) we see that

$$k_{11} \equiv \frac{q}{\sigma_\theta^2}. \tag{C.10}$$

The Kalman gain matrix for the untransformed predetermined variables, K, is finally given by

$$K = L^{-1}\bar{K} = \begin{bmatrix} 1 & 0 \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ \kappa k_{11} & \kappa k_{12} + 1 \end{bmatrix}. \tag{C.11}$$

It remains to show the limit of K when $\sigma_\theta^2 \to \infty$, that is, when $\tilde{y}_t$ becomes an unboundedly noisy indicator of $\bar{y}_t$. We divide (C.5) by σ_θ^2 and observe in (C.6)–(C.8) that

$$\begin{aligned} \frac{a}{\sigma_\theta^2} &\to \tilde{a} \equiv \kappa^2(\rho - \gamma)^2 > 0, \\ \frac{b}{\sigma_\theta^2} &\to \tilde{b} \equiv \kappa^2(1 - \rho^2)\sigma_\eta^2 + (1 - \gamma^2)\sigma_\varepsilon^2 > 0, \\ \frac{c}{\sigma_\theta^2} &\to \tilde{c} \equiv -\sigma_\eta^2\sigma_\varepsilon^2 > 0, \end{aligned}$$

when $\sigma_\theta^2 \to \infty$. It follows that $q \to \tilde{q} > 0$, where $\tilde{q}$ is bounded. Thus, from (C.10) follows that $k_{11} \to 0$.

D The Kalman filter

As a convenient reference, we restate the relevant expressions for the Kalman filter (see Harvey [12] and [13]) in our notation. Let the *measurement* and *transition* equations be, respectively,

$$\begin{aligned} Z_t &= LX_t + v_t, \\ X_{t+1} &= TX_t + u_{t+1}, \end{aligned}$$

where $\mathrm{E}[u_t v_s'] = 0$ for all t and s. Define the covariance matrices of the one-period-ahead and within-period prediction errors by

$$P_{t|t-1} \equiv \mathrm{E}[(X_t - X_{t|t-1})(X_t - X_{t|t-1})'],$$

$$P_{t|t} \equiv \mathrm{E}[(X_t - X_{t|t})(X_t - X_{t|t})'].$$

The covariance matrix of the innovations, $Z_t - Z_{t|t-1}$, fulfills

$$\mathrm{E}[(Z_t - Z_{t|t-1})(Z_t - Z_{t|t-1})'] = LP_{t|t-1}L' + \Sigma_{vv}.$$

The *prediction* equations are

$$\begin{aligned} X_{t|t-1} &= TX_{t-1|t-1}, \\ P_{t|t-1} &= TP_{t-1|t-1}T' + \Sigma_{uu}, \end{aligned}$$

and the *updating* equations are

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + K_t(Z_t - LX_{t|t-1}), \\ K_t &\equiv P_{t|t-1}L'(LP_{t|t-1}L' + \Sigma_{vv})^{-1}, \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1}L'(LP_{t|t-1}L' + \Sigma_{vv})^{-1}LP_{t|t-1}. \end{aligned}$$

In a steady state, we have

$$\begin{aligned} P_{t|t-1} &= P, \\ P_{t|t} &= P - PL'(LPL' + \Sigma_{vv})^{-1}LP, \\ K_t &= K, \\ K &= PL'(LPL' + \Sigma_{vv})^{-1}, \\ P &= T[P - PL'(LPL' + \Sigma_{vv})^{-1}LP]T' + \Sigma_{uu}. \end{aligned}$$

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