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Aggregate uncertainty, HANK, and the ZLB

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Abstract

We propose a novel methodology for solving Heterogeneous Agents New Keynesian (HANK) models with *aggregate uncertainty* and the Zero Lower Bound (ZLB) on nominal interest rates. Our efficient solution strategy combines the sequence-state Jacobian methodology in Auclert et al. (2021) with a tractable structure for aggregate uncertainty by means of a two-regimes shock structure. We apply the method to a simple HANK model to show that: 1) in the presence of aggregate non-linearities such as the ZLB, a dichotomy emerges between the aggregate impulse responses under aggregate uncertainty against the deterministic case; 2) aggregate uncertainty amplifies downturns at the ZLB, and household heterogeneity increases the strength of this amplification; 3) the effects of forward guidance are stronger when there is aggregate uncertainty.

Keywords: Monetary Policy, New-Keynesian Models, Liquidity Traps, Zero Lower Bound, Computational Methods.

JEL Classification: D14, E44, E52, E58

Non-Technical Summary

Recessions are often characterized by increases in aggregate uncertainty and by low nominal interest rates. In this paper, we investigate the macroeconomic interactions between aggregate uncertainty and earnings risk during recessionary episodes when the monetary authority is constrained by an effective lower bound on nominal rates. We do so in the context of a standard Heterogeneous Agent New Keynesian model, usually referred to as HANK. The literature that studies the link between uncertainty at the macro and micro levels is still at its dawn, particularly because solving models that fully account for the totality of nonlinear interactions is very challenging. We develop a new methodology for efficiently solving and simulating heterogeneous-agents' models with nonlinear dynamics (e.g., connected to the presence of a lower bound on policy interest rates) and displaying aggregate uncertainty. The methodology is applied to a HANK model to evaluate the macroeconomic effects of interactions between heterogeneity, non-linearity, and uncertainty.

The methodology is sufficiently general that it can be applied to state-of-the-art HANK models. However, for exposition, we illustrate it in a model with the following characteristics: (i) household earnings are subject to idiosyncratic shocks and (ii) they can partially insure through a riskless asset; (iii) prices of goods and services are subject to nominal rigidities, giving rise to a Phillips Curve; (iv) the discount factor, common to all consumers, is influenced by exogenous disturbances, and is subject to uncertainty (v), and the central bank sets the policy rate through a Taylor Rule (vi) subject to the lower bound constraint. We solve the model calibrated to the United States.

Our solution method is made possible by reducing the number of possible contingencies – paths the economy can follow – to a finite number. In particular, we assume that the economy is initially subjected to a recessionary shock that, in every quarter, can either dissipate or persist in the bad state. In the former case, the economy eventually returns to its initial equilibrium, in a deterministic fashion. In the latter case, the uncertainty structure is repeated in the next quarter: the shock will revert with a certain probability. Because we only allow the shock to remain in a bad state for a (large) number of periods, after which it reverts with certainty, the economy can only follow a finite number of paths. This way we can fully account for the non-linearities stemming from aggregate uncertainty, while preserving tractability. Finally, we explain how to combine a series of state-of-the-art HANK model solution techniques to find the equilibrium along all the possible paths.

We further investigate the dynamics arising from the interplay of aggregate uncertainty and household heterogeneity. Our analysis reveals that when official interest rates reach the lower bound, the presence of aggregate uncertainty amplifies the volatility of economic activity. Specifically, in our baseline model, we observe that this mechanism leads to a substantial increase in output loss compared to the scenario without aggregate uncertainty, more than doubling the negative impact. Moreover, it extends the duration during which nominal rates remain constrained at their lower bound.

We then investigate the interaction between aggregate and idiosyncratic uncertainties. We repeat the previous exercise in an economy populated by a representative agent, commonly referred to as a RANK economy. In this case, the loss in GDP is about 60 percent higher relative to when there is no uncertainty. Taken together with our first result, this shows that heterogeneity among households amplifies the effect of aggregate uncertainty when policy rates are at the lower bound. In other words, it indicates that when policy rates are low, aggregate uncertainty is an important factor for individual choices, including those related to precautionary savings. Our final exercise concerns the role of forward guidance on the policy rate. We find that this policy can be even more effective when we consider a HANK model with aggregate uncertainty, as opposed to its deterministic counterpart, precisely because it is a powerful tool in undoing the strong interactions between aggregate uncertainty and household heterogeneity.

1 Introduction

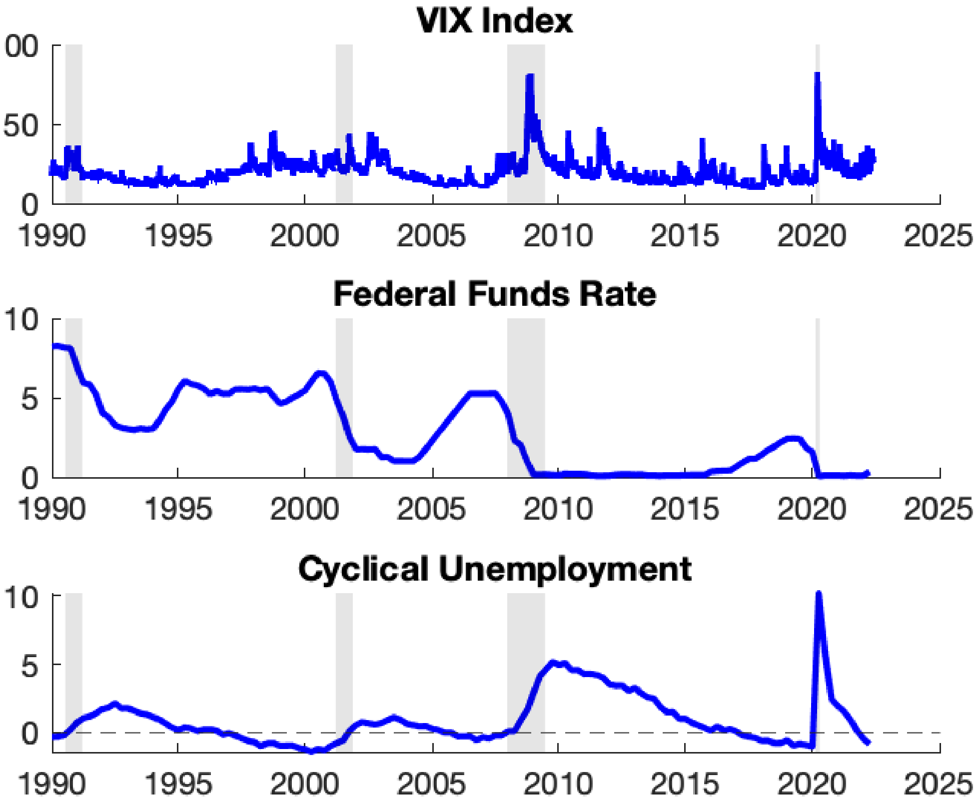
The last two recessions in the US economic history were characterized by: 1) a dramatic spike in aggregate uncertainty; 2) a sharp increase in the unemployment rate above its natural level; and 3) the collapse of monetary policy interest rates to the Zero Lower Bound (ZLB). Figure 1 illustrates these developments by reporting the VIX Index, the federal funds rate, and the difference between unemployment rate and its natural level for the US economy from 1990 to present. In this paper we investigate the macroeconomic interactions between aggregate uncertainty, idiosyncratic risk, and the ZLB, by studying a standard Heterogeneous Agents New Keynesian (HANK) model. We propose a novel solution strategy that allows us to efficiently simulate model economies with complex household heterogeneity structures, aggregate occasionally binding constraints, and a tractable structure of aggregate risk.

The literature has well documented that measures of aggregate uncertainty (see Bloom et al. (2018) or Bloom (2014) for a survey) and of idiosyncratic income risk increase during recessions (see Guvenen, Ozkan and Song (2014) and Shimer (2005) among others). At the same time, it has been shown, both theoretically and empirically, that there are strong interactions between aggregate uncertainty and the ZLB (see for instance Basu and Bundick (2016), Basu and Bundick (2017), Caggiano, Castelnuovo and Pellegrino (2017)).

There is little work, however, in understanding the interactions between the ZLB, idiosyncratic risk, and their policy implications. The literature that studies the interplay between uncertainty at the macro and micro levels is still at its dawn, particularly because solving models that fully account for the totality of non-linear interactions is very challenging. Recent developments in this direction are Fernández-Villaverde et al. (2023) and Kase, Melosi and Rottner (2022), who use neural networks techniques, and Schaab (2020), who develops an adaptive grid methodology.

The main contribution of this paper is to provide a novel solution methodology to heterogeneous agents models with aggregate uncertainty. Our approach allows for rich heterogeneity at micro level and its efficiency grants a large flexibility. We investigate if aggregate uncertainty can have economically relevant effects in amplifying precautionary saving behavior, especially when the economy is up against aggregate non-linearities such as the ZLB. The methodology can be easily used in the presence of aggregate non-linearities other than the ZLB. For instance, non-linear Phillips curves in the presence of inflationary shocks (see Benigno and

Figure 1: VIX Index, Federal Funds Rate, and Cyclical Unemployment Rate



Notes: The figure reports the VIX index, the federal funds rate, and difference between unemployment rate and its natural level for the US economy from 1990 to present. Shaded areas indicate recessions as indicated by NBER. Source: FRED, CBOE, CBO.

Eggertsson (2023) or Gitti (2023)), downward wage rigidities (see Eggertsson, Mehrotra and Robbins (2019)), and aggregate financial constraints among others.

We study a HANK model with nominal interest rates subject to the ZLB and aggregate uncertainty.[1] The computational power required to solve those models is typically large as the combination of multiple idiosyncratic and aggregate states can easily lead to intractability due to the curse of dimensionality. Our first contribution is to develop a fast simulation strategy that allows us to efficiently simulate an economy that features simultaneously: 1) aggregate uncertainty; 2) household heterogeneity and idiosyncratic risk; and 3) the ZLB to nominal interest rates. We consider a simple notion of aggregate uncertainty in the form a Two-states

[1] HANK models with the ZLB have been studied in the context of deterministic shocks. See, for instance Guerrieri and Lorenzoni (2017) and McKay, Nakamura and Steinsson (2016).

Markov process in the spirit of Eggertsson et al. (2021). This allows us to extend the existing solution strategy of Auclert et al. (2021) and work around the curse of dimensionality in the aggregates. Our strategy permits to retain rich household heterogeneity in both income and wealth without the necessity to assume deterministic dynamics in the aggregate economy (i.e. agents do not know with certainty the future). We calibrate the model to U.S. macro (the output loss and inflation during the Great Recession) and micro data, and study counterfactual scenarios to establish three novel results.

First, average dynamics, i.e. the Impulse Response Functions (IRF), are amplified by aggregate uncertainty if the economy is up against the ZLB. We begin by showing that, absent the ZLB, the average effects of our Two-states Markov shock (stochastic shock) are essentially identical to the deterministic effects of the average of the same shock (deterministic shock). To illustrate our result, we then consider the same comparison in the presence of the ZLB. We find large differences: the aggregate uncertainty exacerbates the recession and produces a longer expected duration of the liquidity trap. We quantify this difference by considering the discounted IRF of output and find that the Two-states process exacerbates the output loss by more than 120% relative to the deterministic case.

The second result of our paper concerns the interactions between aggregate uncertainty and idiosyncratic risk. To establish it, we begin by considering a counterfactual Representative Agent economy (RANK) that matches the IRFs obtained in our first exercise, in the absence of the ZLB. We then repeat the first exercise by introducing the ZLB. We find that in the RANK economy at the ZLB, the output loss is "only" 64% larger under aggregate uncertainty when compared to the perfect foresight scenario. We then conclude that household heterogeneity amplifies the effect of aggregate uncertainty at the ZLB. This result suggests that in times of low nominal rates, risk at the aggregate level matters quantitatively for individual choices, in particular for precautionary saving motives.

Our third result concerns unconventional monetary policy evaluations. In our model, we consider the effects of forward guidance and find that it is more effective under the stochastic shock against the deterministic scenario.

The structure of the paper is as follows. Section 2 introduces a simple New Keynesian model that highlights some of the interactions between aggregate uncertainty, idiosyncratic risk, and the ZLB. Section 3 describes the quantitative model, specifies the notion of aggregate uncertainty that we consider, and reports the calibration we use. Section 4 explains the solution strategy. Section 5 compares impulse response functions of models under the stochastic shock against the deterministic counterpart with or without the ZLB, and establishes our first two results. Section 6 studies forward guidance and compares the effects under the stochastic shock and the deterministic one. Section 7 concludes.

2 A Simple Model

This section describes a simple model that we use to illustrate some of the interactions between a tractable stochastic shock, non-linearities such as the ZLB, and idiosyncratic risk. Aggregate uncertainty has a simple Two-states structure and is confined to time 1. The simple model features a stylized structure of household heterogeneity as in Campbell and Mankiw (1989), and Bilbiie (2008).[2] All of these features are present in a richer way in the quantitative model as in section 3.

2.1 Simple Model - Environment

The economy is populated by infinitely lived households who make standard intertemporal consumption-savings decisions to maximize their expected lifetime CRRA utility, with discount factor β_t and relative risk aversion coefficient σ. Every period, households are exogenously assigned to one of two types, constrained c or unconstrained u, according to a fixed transition matrix Q. In the first case, households do not have access to financial markets and consume all of their income. In the second case, households can take a non-negative position in the liquid bond that pays a riskless interest rate. Households incomes are based on their type and are a constant fraction of total income. Liquid bonds are in zero supply. Prices are fully rigid. The economy is closed with a central bank that chooses the gross nominal interest rate R_t following a simple interest rate rule that reacts to output Y_t and is subject to the ZLB.

[2] Other examples of works that belong to this two agents literature are Benigno, Eggertsson and Romei (2020), Bilbiie (2020), Debortoli and Gali (2018), Eggertsson and Krugman (2012), Gali, Lopez-Salido and Valles (2007), Hansen, Lin and Mano (2020), among others.

2.2 Simple Model - Equilibrium

The equilibrium at any point in time is characterized by 2 equilibrium conditions, an aggregate Euler equation and the interest rate rule:

$$
\begin{aligned}
Y_t^{-\sigma} &= \beta_t R_t z_u^{\sigma}\left\{p\left(z_u^{-\sigma}\right)+(1-p)\left[\left(\frac{1-\lambda z_u}{1-\lambda}\right)^{-\sigma}\right]\right\}\mathbb{E}_t Y_{t+1}{}^{-\sigma} = \frac{\beta_t R_t}{\beta R_{ss}}\mathbb{E}_t Y_{t+1}{}^{-\sigma}, && (1)\\
R_t &= \max\left\{\underline{R}, R_{ss} Y_t{}^{\phi}\right\}, && (2)
\end{aligned}
$$

where $\mathbb{E}_t$ is the expectation operator, z_u is the fraction of total income that constitutes the income of an unconstrained household, p is the probability of remaining unconstrained in the next period conditional on being unconstrained currently, λ is the stationary mass of unconstrained households, $\underline{R}$ is a lower bound to gross nominal rates, ϕ governs the reactivity of the central bank and is assumed to satisfy the Taylor principle, β is the steady state value of the discount factor, and $R_{ss} = \frac{1}{\beta\left\{p\left(z_u^{-\sigma}\right)+(1-p)\left[\left(\frac{1-\lambda z_u}{1-\lambda}\right)^{-\sigma}\right]\right\}}$ is the steady state gross interest rate.[3]

For a given value of expected future marginal utility $\mathbb{E}_t Y_{t+1}^{-\sigma}$, the solution is as follows:

$$
Y_t = \begin{cases} \left(\frac{\beta_t}{\beta}\mathbb{E}_t Y_{t+1}{}^{-\sigma}\right)^{-\frac{1}{\sigma+\phi}} & \text{if } \beta_t \leq \beta\left(\frac{R_{ss}}{\underline{R}}\right)^{\frac{\sigma+\phi}{\phi}}\left(\mathbb{E}_t Y_{t+1}^{-\sigma}\right)^{-1} \\ \left(\frac{\beta_t}{\beta}\frac{\underline{R}}{R_{ss}}\mathbb{E}_t Y_{t+1}^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{otherwise} \end{cases}, \tag{3}
$$

$$
R_t = \begin{cases} R_{ss}\left(\frac{\beta_t}{\beta}\mathbb{E}_t Y_{t+1}{}^{-\sigma}\right)^{-\frac{\phi}{\sigma+\phi}} & \text{if } \beta_t \leq \beta\left(\frac{R_{ss}}{\underline{R}}\right)^{\frac{\sigma+\phi}{\phi}}\left(\mathbb{E}_t Y_{t+1}{}^{-\sigma}\right)^{-1} \\ \underline{R} & \text{otherwise} \end{cases}. \tag{4}
$$

In particular, focusing on output, we will have a piece-wise non linear function $Y_t = f\left(\mathbb{E}_t Y_{t+1}^{-\sigma}, \beta_t \middle| \beta, \sigma, \phi, \underline{R}, R_{ss}\right)$. Clearly, the higher the expected future marginal utility (or the higher the discount factor) the larger the current recession. We will therefore first focus on the effects of aggregate uncertainty on the expected future marginal utility and then move to the actual effects on current output. Notice that we can ignore the lower bound to interest rates as long as it is set "low enough".

2.3 A Stochastic Shock

We consider the following chain of events. The economy begins at $t = 0$ where households enter with no wealth and the respective masses across different types are the stationary ones

[3] The notation $\underline{R}$ generalizes the possibility of an effective lower bound (ELB) as opposed to the ZLB.

(λ and $1-\lambda$ as implied by the transition matrix Q).[4] Households know their discount factor $\beta_0 \geq \beta$ for $t=0$ and $\beta_t = \beta$ for any $t \geq 1$. They also know that there is a probability μ that the discount factor at $t=1$ will be $\beta_1 = \beta_{1L} \geq \beta$, $\beta_1 = \beta$ otherwise.[5]

The assumption on the stochastic structure allows us to divide into only two possible paths that the economy can follow in the aggregate: 1) the history in which at $t=1$ the discount factor is back to its stationary level, $\beta_1 = \beta$, or 2) the one in which agents are more patient with $\beta_1 = \beta_{1L}$. We will refer to the first case as *contingency 1*, indicating the time at which the exogenous discount factor, or shock, dissipates. Similarly, the second history is denoted as *contingency 2* because the discount factor goes back to its steady state level at $t=2$.

2.4 Solution - Outside the ZLB

The model is purely forward looking and simple enough that the solution at $t=2$ is the steady state, meaning that, in both contingencies,

$$Y_2 = Y_{ss} = 1,$$
$$R_2 = R_{ss}.$$

Now consider the solution at a generic $t=1$ under the assumption that the lower bound on interest rates does not exist (equivalently $\underline{R} = -\infty$). One can show that solution will be a function of the discount factor at $t=1$ only. We write it in terms of marginal utility:

$$Y_1^{-\sigma} = \left(\frac{\beta_1}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}}.$$

[4]Notice that in this simple model, the fraction λ is constant and exogenous. We will relax this assumption in section 3, where the fraction of households that are financially constrained will be time varying and endogenously determined.

[5]This implies that a household who is unconstrained at $t=0$, knows that at $t=1$ she will be unconstrained with a large discount factor with probability μp, she will be unconstrained with a steady state discount factor with probability $(1-\mu)p$, knows that she will be constrained with a large discount factor with probability $\mu(1-p)$, and constrained with a steady state discount factor otherwise.

Let us now consider the solution at time $t = 0$. One can show that solution can be written as follows:

$$Y_0^{-\sigma} = \left(\frac{\beta_0}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}} \left[\mu Y_{1L}^{-\sigma} + (1-\mu)\, Y_{ss}^{-\sigma}\right] = \left(\frac{\beta_0}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}} \underbrace{\left[\mu\left(\frac{\beta_{L1}}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}} + (1-\mu)\right]}_{\mathbb{E}_0 Y_1^{-\sigma}}, \tag{5}$$

where Y_{1L} denotes the output at $t = 1$ in *contingency 2*. The top-left panel in figure 2 shows the equilibrium marginal utilities at $t = 1$ as a function of the discount factor β_1. A larger the discount factor leads to a larger marginal utility, in turn implying a larger output loss. The plot also reports the expected future marginal utility on the red dotted line, corresponding to a linear combination between the marginal utilities in the two contingencies.

2.5 The deterministic counterpart

We want to consider the effects of a stochastic shock and compare them to those of a deterministic one. To do that we consider a similar economy whose only difference is the discount factor at $t = 1$ will be β_{1DET} with probability 1. We choose this discount factor so that the effect at $t = 0$ is the same, meaning that the expected future marginal utility is equalized:

$$\mu Y_{1L}^{-\sigma} + (1-\mu)\, Y_{ss}^{-\sigma} = Y_{1DET}^{-\sigma}$$
$$\mu\left(\frac{\beta_{L1}}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}} + (1-\mu) = \left(\frac{\beta_{1DET}}{\beta}\right)^{\frac{\sigma}{\sigma+\phi}},$$

as reported in the top-left panel of figure 2. Obviously, the effects on the equilibrium at $t = 0$ under the stochastic or the deterministic shocks are identical by construction, as shown in the bottom-left panel of figure 2, which relates output Y_0 to the expected future marginal utility $\mathbb{E}_0 Y_1^{-\sigma}$.

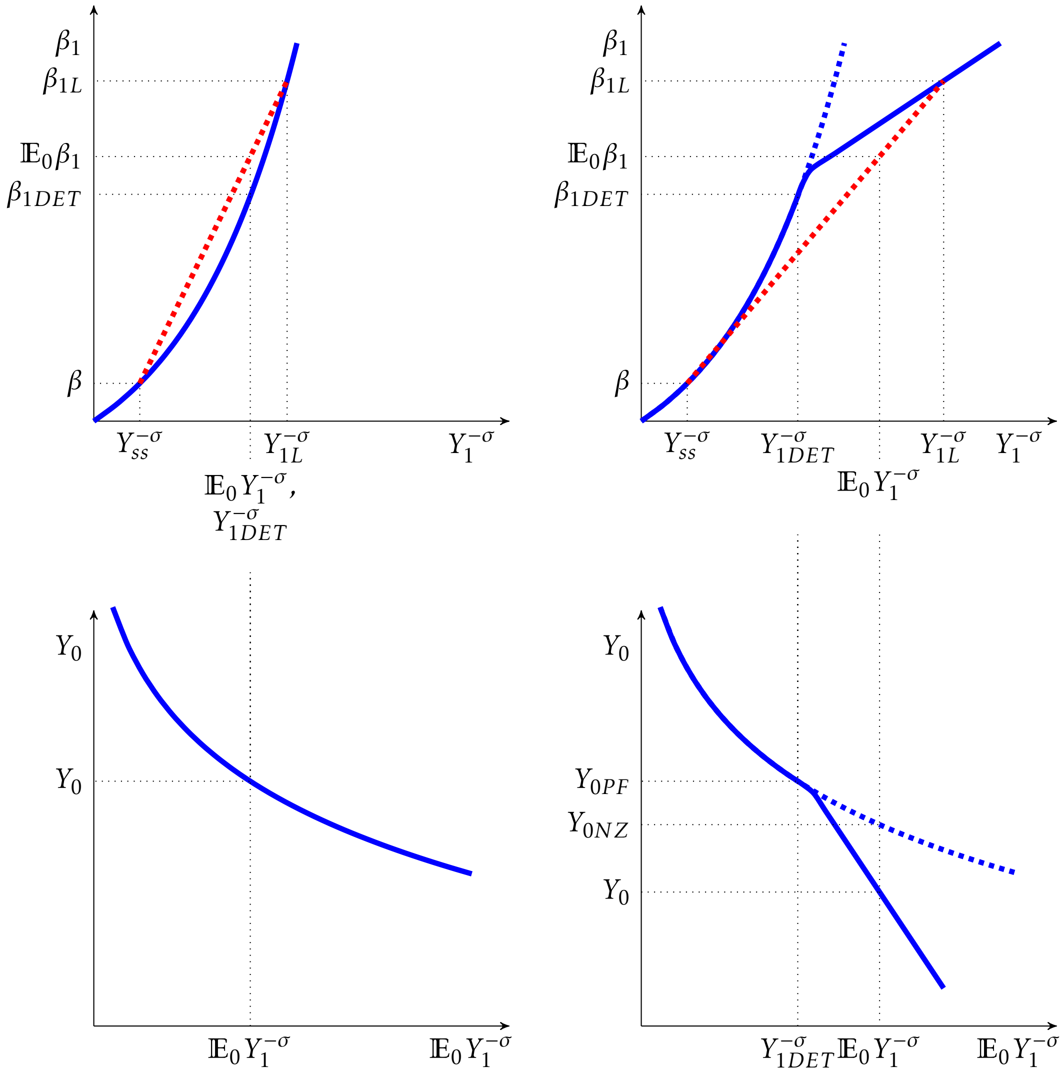
2.6 The Effects of the ZLB

In the previous part of this exercise, we have constructed two economies that differ in terms of shock structure, but yield the same effect on the expected future marginal utility $\mathbb{E}_0 Y_1^{-\sigma}$, hence the same effect on output on impact Y_0.

We now show that the introduction of non linearities such as the ZLB can actually generate a dichotomy between the two economies, implying a meaningful interaction of aggregate

Figure 2: Equilibrium in the Simple Model

(a) Outside the ZLB: $\underline{R} = -\infty$ (b) With the ZLB: $\underline{R} = 1$



Notes: The figure shows the equilibrium of the simple model without the ZLB (left column, $\underline{R} = -\infty$) and with the ZLB (right column, $\underline{R} = 1$). The top panels report the equilibria at $t = 1$. The blue solid lines show the relationship between the discount factor β_1 on the y-axis and the corresponding marginal utility $Y_1^{-\sigma}$ on the x-axis. They also report the corresponding expected value $\mathbb{E}_0 Y_1^{-\sigma}$, obtained with a linear combination along the red dotted line. The blue dotted line on the top-right panel is reported for comparison. The bottom panels report the equilibria at $t = 0$. The blue solid lines show the relationship between output Y_0 on the y-axis and expected future marginal utility $\mathbb{E}_0 Y_1^{-\sigma}$ on the x-axis. The blue dotted line on the top-right panel is reported for comparison.

uncertainty and the ZLB itself. To do this, we again consider the effects on expected future marginal utility, using the same shocks used in the previous exercise but setting $\underline{R} = 1$.[6] One can show that once the ZLB is hit, the slope of equation (3) becomes steeper (the slope looks flatter as the axis are inverted), as shown in the top-right panel of figure 2.

In the most interesting scenario, the ZLB happens to bind at $t = 1$ in contingency 2, which dramatically increases the expected future marginal utility. On the contrary, the equilibrium in the deterministic counterpart is unaffected.[7] As a consequence, the equilibrium at $t = 0$ will be significantly different between the stochastic and deterministic case, as shown in the bottom-right panel of figure 2. The plot reports three possible outcomes at $t = 0$: 1) the deterministic one Y_{0PF}; 2) the stochastic one in which the ZLB does not bind at $t = 0$, Y_{0NZ}; and 3) the stochastic one in which the ZLB does bind at $t = 0$, Y_0.

The fact that introducing the ZLB could cause one of two outcomes at $t = 0$ depending on the severity of the shock, stresses that there is a possible compounding chain that amplifies the overall effects of a stochastic shock. First, the stochastic shock interacts with the ZLB by significantly affecting the expected future marginal utility. Second, if this effect happens to be large enough, it also leads the economy to the ZLB at $t = 0$, creating a sort of cascading effect.

2.7 The Role of Household Heterogeneity

The mechanism that we explained shows that there exists an economically relevant interaction between the ZLB and the aggregate uncertainty as defined in the simple example. However, our discussion has not yet touched upon the role of household heterogeneity.

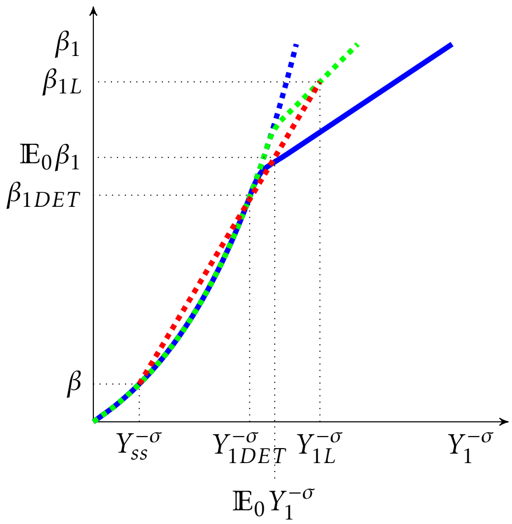
There are two main channels through which household heterogeneity affects our result. First of all, in the presence of idiosyncratic risk, households display precautionary savings behavior decreasing the interest rate level in the steady state. In this simple model, this is equivalent to a decrease in R_{ss}. As the interest rate gets closer to the ZLB, the risk of entering a liquidity trap increases, or in other words, smaller shocks will lead to the ZLB. Conversely, in a model without idiosyncratic risk (the RANK version of the simple model), the threshold level for the discount factor β_1 that triggers the liquidity trap increases.

Second, the presence of idiosyncratic risk steepens the aggregate response once the ZLB

[6] This assumption is convenient for algebraic purposes, but the actual level of the ELB has no impact on the argument we make.

[7] This is one possible scenario. However, one can show that even if the ZLB was to bind in the deterministic economy, the overall effect on expected future marginal utility would be smaller than under the stochastic scenario.

Figure 3: Equilibrium at $t = 1$ in the Simple Model - Effects of Heterogeneity



Notes: The figure shows the equilibrium of the simple model with the ZLB ($\underline{R} = 1$) and the effects of household heterogeneity. The figure reports the equilibrium at $t = 1$ under the RANK model, where idiosyncratic risk is shut down. The green dotted line show the relationship between the discount factor β_1 on the y-axis and the corresponding marginal utility $Y_1^{-\sigma}$ on the x-axis. We also report the corresponding expected value $\mathbb{E}_0 Y_1^{-\sigma}$, obtained with a linear combination along the red dotted line. The blue lines are reported for comparison and correspond to the model with idiosyncratic risk.

is hit thanks to the presence of high marginal propensity to consume agents. In other words, conditional on hitting the ZLB, the marginal effect of a change in the discount factor is smaller under the RANK model as opposed to the model with idiosyncratic risk.

To reiterate, from an economic perspective, these two effects mean that the presence of idiosyncratic risk brings a level effect (by pushing the economy closer to the ZLB) and a marginal effect (by making the economy more reactive once the ZLB is hit). All of these effects are shown in figure 3, which shows how the effect on expected future marginal utility is mitigated in the model without idiosyncratic risk. It then follows that all the cascading effects mentioned before are also attenuated in the RANK version of the simple model. Equivalently, the presence of idiosyncratic risk contributes in amplifying the interaction between aggregate uncertainty and ZLB.

3 A HANK Model with Aggregate Uncertainty

In the previous section we used a simple model to establish the main mechanisms at the origin of the interactions between aggregate uncertainty, idiosyncratic risk, and the ZLB. This section describes a richer HANK model that we use to illustrate our solution strategy and results, formally explains the notion of aggregate uncertainty that we use, and introduces the notational convention that we use in later sections.

3.1 Model

The model economy is populated by a continuum of infinitely-lived households, intermediate producers, a final good aggregator, and the government sector that comprises the fiscal and monetary authorities.

Households There is a continuum of measure 1 of infinitely lived households i who maximize their discounted lifetime utility (of constant relative risk aversion form) from consumption, $\mathbb{E}_t \sum_{s=t}^{\infty} \left(\prod_{j=t}^{s} \beta_j \right) \frac{c_{it}^{1-\sigma}}{1-\sigma}$. Households inelastically supply the labor amount n_t required by firms and receive labor income $z_{it} w_t n_t$, where w_t is the real wage rate per efficient hour and z_{it} is the household idiosyncratic productivity, which can take one of n_z values. The matrix $Q_t(\cdot)$ disciplines the transition between idiosyncratic productivity states.[8] The productivity level also determines dividend payments $z_{it} d_t$ and taxation $z_{it} t_t$, where d_t and t_t are aggregate profits from the firm sector and aggregate taxation from the fiscal authority. Those assumptions imply that the income flow is proportional to aggregate output Y_t net of taxation, $z_{it}(Y_t - t_t)$. Households can save in nominal riskless bonds a_{it}, whose price is the (inverse of) risk-free gross nominal interest rate R_t and are subject to a borrowing constraint $\underline{a}$.

Consider a household with idiosyncratic state z_{it} and initial savings a_{it-1}, whose real value is depreciated by current inflation Π_t. The maximization problem is represented by the following value function.

$$
\begin{aligned}
V_t(z_{it}, a_{it-1}) = \max_{c_{it}, a_{it}} & \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_t \mathbb{E}_t V_{t+1}(z_{it+1}, a_t) \\
\text{s.t. } & c_{it} + \frac{a_{it}}{R_t} = \frac{a_{it-1}}{\Pi_t} + z_{it}(Y_t - t_t) \\
& a_{it} \geq \underline{a}
\end{aligned} \tag{6}
$$

where the time dependence of the value function captures all the variations in prices (Y_t, Π_t,

[8] We consider a special case for the transition probabilities: we assume a time-invariant transition matrix which implies that individual risk is acyclical. Equivalently, the variance of log-incomes is time invariant in our model. This is a rather conservative assumption since it does not intrinsically generate recessions amplifications. Empirically, idiosyncratic risk is procyclical (see Schaab (2020) for some empirical evidence on the search and matching probabilities over the business cycle).

R_t, and t_t) and in aggregate exogenous shocks (β_t). The expectation operator $\mathbb{E}_t$ embeds the uncertainty at both the aggregate and idiosyncratic level. The aggregate shock process, which corresponds to exogenous movements in the discount factor, will be specified later.

The optimization problem yields the standard Euler equation optimality condition and individual asset demand, which we write in the individual state space.

$$\frac{c_t(z_{it}, a_{it-1})^{-\sigma}}{\Pi_t} \geq \beta_t \frac{R_t}{\Pi_t} \mathbb{E}_t \left[c_{t+1}(z_{it}, a_{it-1})^{-\sigma} \frac{1}{\Pi_{t+1}} \right] \tag{7}$$

$$a_t(z_{it}, a_{it-1}) = R_t \left[\frac{a_{it-1}}{\Pi_t} + z_{it}(Y_t - t_t) - c_t(z_{it}, a_{it-1}) \right] \tag{8}$$

Let us define the distribution $D_t(z_{it}, a_{it-1})$ over the individual states space at the beginning of the period. The household problem yields two aggregate objects, consumption and asset demand, defined as follows:

$$C_t \equiv \int c_t(z_{it}, a_{it-1})\, dD_t,$$
$$A_t \equiv \int a_t(z_{it}, a_{it-1})\, dD_t.$$

Supply Side The supply side follows the New Keynesian tradition with a continuum of intermediate producers with monopolistic power and quadratic price adjustment costs, a competitive final good producer, and labor supply entity ("union") that decides the labor to be supplied based on a wage schedule that resembles that of a Representative Agent. We assume that there is a sales subsidy to eliminate monopolistic distortions in the intermediate sector, and that they are rebated lump-sum to the firms.

The supply side conveniently aggregates to the New Keynesian Phillips curve, where the labor supply schedule has a Frisch elasticity ω, and the slope parameter κ is inversely related to the degree of price adjustment. The reader can refer to the appendix in Lin (2020) for more details on the supply side.

$$\left(\Pi_t - \overline{\Pi}\right)\Pi_t = \mathbb{E}_t \beta_t \left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma} \left(\Pi_{t+1} - \overline{\Pi}\right)\Pi_{t+1} + \kappa\left[Y_t^{\omega+\sigma} - 1\right] \tag{9}$$

Fiscal Policy The government imposes lump-sum taxes to ensure a balanced budget and to hold real debt to a fixed amount.[9]

$$t_t + \frac{b_t}{R_t} = \frac{b_{t-1}}{\Pi_t} \tag{10}$$

$$b_t = \overline{b} \tag{11}$$

Monetary Policy The Central Bank follows a standard Taylor rule, reacting to deviations of inflation and output from their respective steady state values, $\overline{\Pi}$ and $\overline{Y}$, and is subject to the ZLB.

$$R_t = \max\left\{\underline{R}, \overline{R}\left(\frac{\Pi_t}{\overline{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\overline{Y}}\right)^{\phi_y}\right\} \tag{12}$$

Market Clearing Conditions Assets and goods markets clearing conditions follow.

$$b_t = A_t \tag{13}$$

$$C_t = Y_t \tag{14}$$

3.2 Equilibrium (generic)

An equilibrium in this economy is represented by a series of stochastic processes for the aggregate variables $X_t = \{Y_t, \Pi_t, b_t, t_t, R_t\}$, a series of stochastic functions for the individual choices $g_t(z_{it}, a_{it-1}) = \{c_t(z_{it}, a_{it-1}), a_t(z_{it}, a_{it-1})\}$, given an initial distribution D_0 and a stochastic process for the discount factor β_t, such that:

- individual policy functions solve the household maximization problem (6);
- the distribution law of motion is consistent with individual policy functions (7) and (8);
- equations (9)-(14) hold at all times.

Traditional solution methodologies for models similar to ours are typically burdensome for several reasons. First, the individual distribution is an infinitely dimensional object that needs

[9] The assumption on fiscal policy can be relaxed with a different fiscal policy but local stability must be ensured. For example, the fiscal policy could follow a tax policy so that $t_t = \overline{t} + \phi_t\left(Y_t - \overline{Y}\right)$, where if $\phi_t > 0$, we have a case with countercyclical fiscal policy.

to be kept track since it is a state variable in the economy.[10] Second, the possibility of *many* different possible trajectories for the economy increases the complexity in the aggregate variables (since one need to consider many possible future realizations for a variable at a certain period) and exacerbates the first problem. Third, the presence of aggregate non-linearities such as the ZLB make standard perturbation inaccurate.

Below we specify the stochastic process for the discount factor β, explain its economic implications, and describe how it allows us to define the equilibrium in a more compact way.

3.3 Aggregate Uncertainty

In this subsection we introduce our notion of aggregate uncertainty, a Two-states Markov process with an absorbing state. We explain how this structure allows us to significantly reduce the severity of the computational burdens described above.

Motivation Most recessions can be viewed as a consequence of an unexpected event whose precise duration is unknown from an ex-ante perspective. Once such shock dissipates, then the economy moves "back on track".[11]

One prominent example is COVID-19. On the onset of the pandemic, the timing frame for the availability of vaccines was far from clear. However, the consensus was that, once a considerable portion of the population would be vaccinated, *normalcy would be restored* and the economy would recover to pre-crisis levels.[12] A second example is the Great Recession, in which uncertainty about the speed of recovery was also high, but there was no disagreement about the fact that the economy would eventually be back on track.

Assumptions We assume that the economy begins at its stationary equilibrium and at time $t = 0$ the discount factor unexpectedly becomes $\beta_L > \beta$. Every period there is a fixed probability that it reverts back to its steady state value.[13] Formally we have the following expression for

[10] There are a few exceptions to this problem. Acharya and Dogra (2020) and Acharya, Challe and Dogra (2023) work around it by assuming CARA utility functions, which allows for linear aggregation of individual policies. There is also a strand of the literature that uses continuous time techniques to work around some of the computational hurdles. See for instance Achdou et al. (2022), Ahn et al. (2018), and Kaplan, Moll and Violante (2018) among others.

[11] In both cases, the uncertainty regarding aggregate dynamics was unprecedented and we argue that our solution strategy is able to capture meaningful economic effects, as opposed to standard techniques.

[12] See for instance https://www.theatlantic.com/family/archive/2020/03/coronavirus-social-distancing-over-back-to-normal/608752/.

[13] This aggregate shock structure is not novel. Eggertsson and Woodford (2003) and Eggertsson et al. (2021) are other examples that have similar settings. A more structured shock process (a convolution of an AR(1) process

the discount factor:

$$\beta_t = \begin{cases} \beta & \text{if } \beta_{t-1} = \beta \\ \beta & \text{w.p. } (1-\mu) \text{ if } \beta_{t-1} = \beta_L \\ \beta_L & \text{w.p. } \mu \text{ if } \beta_{t-1} = \beta_L \end{cases}, \tag{15}$$

where $\mu \in [0,1]$ is the reversal probability from the crisis state to the normal times state. Figure 4 represents the shock in a graphical way. Each blue line represents one possible profile for the discount factor, and the thickness of the line corresponds to the unconditional probability of said profile. The red line represents the unconditional expectation.[14] The figure also reports a black dashed line which represents a deterministic counterpart to the whole stochastic structure.[15] It is worth mentioning that the stochastic structure in our setup is essentially different from the standard shock structure present in most DSGE models, where every period there is a random disturbance drawn from a normal distribution, as in Kase, Melosi and Rottner (2022), Fernández-Villaverde et al. (2023), and Schaab (2020). In those setups, there is a notion of aggregate risk in the long-run, while in our case, shocks will dissipate leading the economy to a deterministic steady state.

Notation and Terminology We refer to a *contingency* as the time τ when the shock switches back to its steady state value as well as the equilibrium dynamics of the economy following such event.[16] As an example, if the discount factor in our model switches back to its steady state value β at time 8, the aggregate economic trajectory following this event is what we refer to as contingency 8, i.e. $\tau = 8$.

We use the notation x_t^τ to indicate the value of variable x, at time t, under contingency τ, and x_t to indicate the value of variable x, at time t, when the shock has *not yet* reverted.[17] Those are essentially different economic objects. For a concrete example, consider inflation at time $t = 2$. In our setup, there are three distinct "inflation-at-time-2" economic objects that are

together with a Two-states Markov process) can be found in Lin (2020).

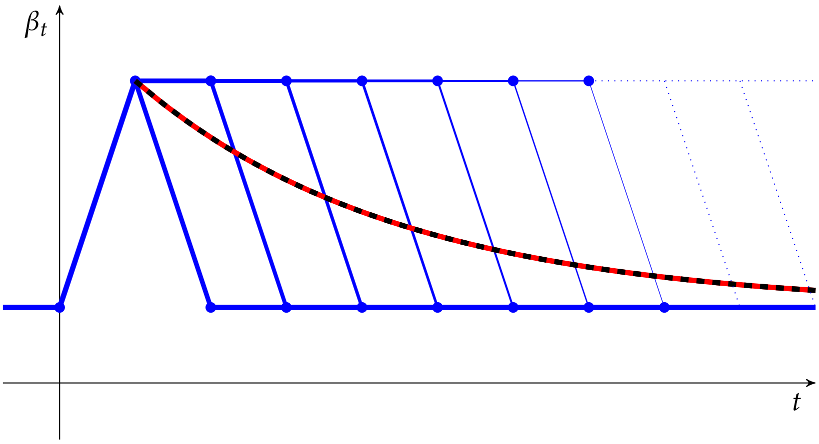
[14] To construct the unconditional expectation we weight each possible profile by its corresponding probability. This is also what applies to households, that is, they have rational expectations. In principle, our solution strategy allows us to depart from rational expectations but only in the particular way in which agents just apply a different probability than μ to the aggregate process.

[15] In the case of the shock, the deterministic counterpart coincides with the unconditional expectation. However, as the reader will later see, this is not necessarily true for endogenous variables.

[16] Note that there is no such thing as contingency 0.

[17] Note that it must be that $t \geq \tau$.

Figure 4: Graphical Representation of Shock



Notes: The figure represents the shock structure and its average (red line). The black dashed line being the deterministic counterpart.

relevant: inflation at time 2 in contingency 1, Π_2^1; inflation at time 2 in contingency 2, Π_2^2; and inflation at time 2 in any contingency larger than 2, Π_2.

We also define the collection of aggregate prices at time t before the shock regime reverts as $X_t \equiv \{Y_t, \Pi_t, R_t, t_t, b_t\}$ and the same set of objects in contingency τ at time t as $X_t^\tau \equiv \{Y_t^\tau, \Pi_t^\tau, R_t^\tau, t_t^\tau, b_t^\tau\}$.

Finally, from now on we use the same notation to denote the distribution over the idiosyncratic states at the beginning of the period t in contingency τ, D_t^τ, and at the beginning of period t, when the shock has not yet reverted, D_t. The same holds for the value functions, V_t^τ and V_t.

Implications of the Two-State Shock Structure The shock structure assumption has three main implications in our model. First, the assumption of the Two-states structure significantly reduces the numerosity of the economic objects. Specifically, it implies that there are only $t+1$ possible values for a certain aggregate variable at time t.

The second implication concerns the fact that once in a contingency, predetermined variables such as the initial distribution D_τ^τ, or past aggregate variables $X_{\tau-1}$, can be taken as given and the economy becomes deterministic in the aggregate. We can then adapt some of the techniques introduced by Auclert et al. (2021) to account for different initial conditions (see section 4). The implied gains in computing time are large since this procedure can be easily parallelized.

The third implication of the shocks structure pertains forward-looking variables during the periods in which the shock has not reverted yet, in which case we need to explicitly take the uncertainty into account. The simple structure allows us to write expectations in a compact way. Consider inflation at $t = 1$, from the perspective of $t = 0$. The expectation of one period ahead inflation can be compactly written as $\mu\Pi_1 + (1-\mu)\Pi_1^1$. Now consider the consumption Euler equation (7) at time 0, when households are aware of the uncertainty:

$$\frac{c_0(z, a_{-1})^{-\sigma}}{\Pi_0} \geq \beta_0 \frac{R_0}{\Pi_0} \left\{ (1-\mu) \left[\sum_{z'} Q_{z,z'} c_1^1 (z', a)^{-\sigma} \frac{1}{\Pi_1^1} \right] + \mu \left[\sum_{z'} Q_{z,z'} c_1 (z', a)^{-\sigma} \frac{1}{\Pi_1} \right] \right\},$$

where $Q_{z,z'}$ indicates the probability of moving from productivity level z to z'.

3.4 Calibration

The main calibration we use is summarized in table 1. In the steady state quarterly output is normalized to 1, the annualized inflation rate is set to 2%, and the supply of liquid bonds equals 25% of yearly GDP. The discount factor is set to clear the asset market, $\beta = 0.9805$.

The CRRA utility parameter is set to 1.5 as in Smets and Wouters (2007). The Frisch elasticity is set to $\omega = 1$. We set the monetary policy parameters to standard values, $\phi_\pi = 1.5$ and $\phi_y = 0.125$. The idiosyncratic risk process is taken from McKay, Nakamura and Steinsson (2016).

We calibrate the shock size $\beta_L = 0.993$ and the slope of the New Keynesian Phillips curve ($\kappa(\omega + \sigma)$) to 0.01 to obtain initial output and inflation, in the HANK model with the ZLB, that match those of the Great Recession.[18] The shock reversal parameter μ is taken from Eggertsson et al. (2021).

[18] In a representative agent model, the shock decreases the natural rate by 4.8% on an annualized basis.

Table 1: Calibration

Parameter	Value	Source	Note
σ	1.5	Smets and Wouters (2007)	EIS
β	0.9805	Calibrated	Discount Factor
ω	1	Standard	Frisch elasticity
κ	$\frac{0.01}{\omega+\sigma}$	Calibrated	Phillips Curve
$\overline{\Pi}$	$1.02^{0.25}$	Standard	Inflation target
ϕ_π	1.5	Standard	Monetary Policy
ϕ_y	0.125	Standard	Monetary Policy
z		McKay, Nakamura and Steinsson (2016)	Idiosyncratic Shocks
Q		McKay, Nakamura and Steinsson (2016)	Idiosyncratic Shocks
μ	0.9	Eggertsson et al. (2021)	Switching Probability
β_L	0.993	Calibrated	Shock

Notes: the table reports the calibration used in the paper. See text for more details.

4 Solution Approach

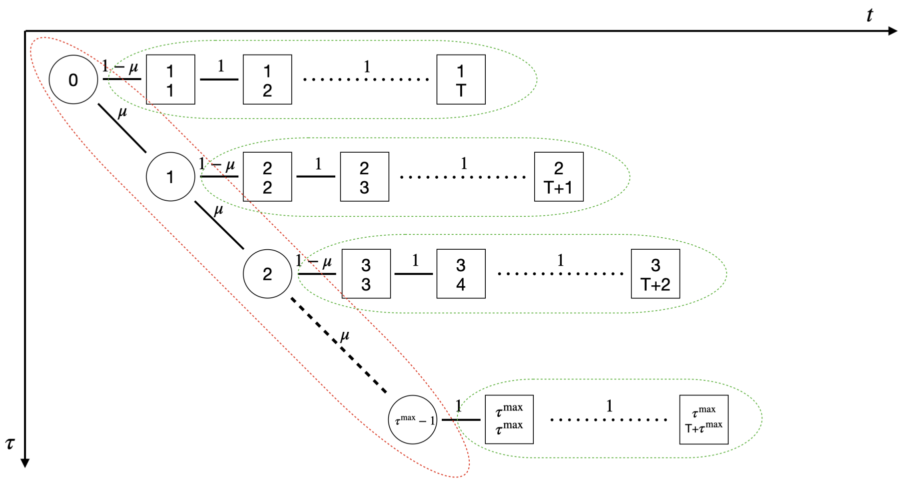
Our solution methodology can also accommodate the presence of non-linearities on the behavior of the (macro) economy. In our applications, the Central Bank is constrained by the ZLB on nominal interest rates. Other applications of our solution algorithm include, but are not limited to, aggregate financial constraints, downward wage rigitidies (see Eggertsson, Mehrotra and Robbins (2019)), or non-linear Phillips curves (see Benigno and Eggertsson (2023), Comin, Johnson and Jones (2023), or Gitti (2023)), which might be of particular relevance in applications studying the recent inflation surge.

Both for exposition and because it is crucial for our solution method, we group the types of paths the economy can take into two: the *Two-state* regime (henceforth **TS**), in which the uncertainty has yet to be resolved, and the set of *perfect-foresight* paths (henceforth **PF**), in which the shock (regime) has already reverted.

Figure 5 illustrates the classification. The TS path is represented by the diagonal line, highlighted in red, whereas the PF paths are highlighted in green. In the diagram, the vertical axes represents the contingency τ, time at which the shock reverts, whereas the horizontal axis indicates actual time t.

The economy is initialized in its steady state, represented in the upper-left node, marked by 0. An unexpected shock then materializes, and, in each period, the economy can revert towards its deterministic path towards the steady-state value with probability $1-\mu$. In case it does not, the economy follows the *diagonal* TS path to the circle marked by 1. Alternatively, if the shock reverts, the economy then moves *horizontally*, entering a deterministic path.

Figure 5: Representation of the Economy under the Two-State Shock Structure



As can be seen in the diagram, we impose two technical, albeit harmless, assumptions to be able to implement our methodology. First, we impose a period $\tau^{\max}$ at which the shock reverts with probability one. And second, we assume that the economy returns to its steady state T periods after the regime reversion.

Before proceeding to the solution methodology, we need to define further notation. Along the TS branch, we define $\mathbb{X}^{TS}$ as the (stacked) vector of equilibrium objects $\{X_t\}_{t=0}^{\tau^{\max}-1}$, $\mathbb{D}^{TS}$ as a matrix made of $\tau^{\max}$ distributions over idiosyncratic states $\{D_t\}_{t=0}^{\tau^{\max}-1}$, and $\mathbb{V}^{TS}$ as a matrix of $\tau^{\max}$ value functions $\{V_t\}_{t=0}^{\tau^{\max}-1}$. For the PF contingencies, we let $\mathbb{X}^{\tau}$, $\mathbb{D}^{\tau}$, and $\mathbb{V}^{\tau}$ respectively represent the set of equilibrium objects $\left(\{X_t^{\tau}\}_{t=\tau}^{T+\tau-1}\right)$, distributions $\left(\{D_t^{\tau}\}_{t=\tau}^{T+\tau-1}\right)$, and value functions $\left(\{V_t^{\tau}\}_{t=\tau}^{T+\tau-1}\right)$ along contingency τ. Furthermore, we denote $\mathbb{X}^{PF}$, $\mathbb{D}^{PF}$, and $\mathbb{V}^{PF}$ as respectively the complete set of the equilibrium objects $\left(\{\mathbb{X}^{\tau}\}_{\tau=1}^{\tau^{\max}}\right)$, distributions $\left(\{\mathbb{D}^{\tau}\}_{\tau=1}^{\tau^{\max}}\right)$, and value functions $\left(\{\mathbb{V}^{\tau}\}_{\tau=1}^{\tau^{\max}}\right)$ along the entire set of PF branches. Lastly, we denote $\mathbb{X}_{ss}^{TS}$ and $\mathbb{X}_{ss}^{PF}$ to be the (stacked) vectors of steady-state values with respective dimension $5\tau^{\max}\times 1$ and $5T\times 1$. Refer to appendix A for a more detailed overview on our notation.

Broad Overview of the Methodology Our solution methodology successively iterates between solutions of the TS and the PF paths until a fixed point of all state variables - including the entire distribution of households over idiosyncratic states - along the diagonal path is achieved. A brief description is provided in algorithm 1 below:

Algorithm 1 *Broad Overview of the Solution Methodology*

1. *Provide a guess for the economy's states at the initial period in each PF path,* $\left\{\{D_\tau^{\tau}\}_{\tau=1}^{\tau^{\max}}, \{X_\tau\}_{\tau=0}^{\tau^{\max}-1}\right\}^n$,

with $n = 0$.

2. *The guess consists of a set of initial conditions in each PF path. Conditional on those, solve for the equilibrium in each of the contingencies. In particular, collect the relevant forward looking variables for each node along the diagonal path, $\left\{\{V_\tau^\tau\}_{\tau=1}^{\tau^{\max}}, \{X_\tau^\tau\}_{\tau=1}^{\tau^{\max}}\right\}$.*

3. *Keeping the forward looking variables fixed, solve for the equilibrium along the TS path. In particular, obtain a new set of initial conditions for each PF path, $\left\{\{D_\tau^\tau\}_{\tau=1}^{\tau^{\max}}, \{X_\tau\}_{\tau=0}^{\tau^{\max}-1}\right\}^{n+1}$.*

4. *If the newly obtained state variables are sufficiently close to the guess, an equilibrium for the economy is found. Otherwise, return to step 2.*

Before delving into details of each of the steps above, we need lay out some notation to illustrate how our economy can be treated as a (large) system of equations to be solved using our novel methodology.

4.1 General Equilibrium in Our Setup

Let $\mathbb{Z}^{\text{TS}} = \{Z_t\}_{t=0}^{\tau^{\max}-1}$ represent the dynamics of exogenous disturbances along the diagonal path. In our baseline model, this is given by $\beta_t = \beta_L$ for $t \in \{0, 1..., \tau^{\max-1}\}$. To simplify our exposition, we have assumed that once the shock reverts the shock is immediately back to its steady-state value.[19] Following Auclert et al. (2021), the general equilibrium in our model can be expressed by the system of equations:

$$\mathbf{F}\left(\mathbb{X}^{\text{TS}}, \mathbb{X}^{\text{PF}}, \mathbb{Z}^{\text{TS}}\right) = 0, \tag{16}$$

In the model laid out in section 3, $\mathbf{F}(\cdot)$ consists of equations (9)-(13) at *each* period of both the TS and the PF paths.

Although we do not necessarily solve the system of equations (16) using a first-order approximation, as we will see below, our methodology does make use of perturbation techniques to solve the system of equations. In particular, it requires the computation of the Jacobians of equilibrium conditions (9)-(13). For the subset of those equations that can be written analytically, these derivatives have an analytical representation and thus their computation is

[19] This is not necessary for our solution strategy to be valid. First, our methodology easily adapts to arbitrary shocks on the diagonal path. Second, it also accommodates arbitrary shock values once the contingency is revealed. In this case, equation (16) would instead read $F\left(\mathbb{X}^{\text{TS}}, \mathbb{X}^{\text{PF}}, \mathbb{Z}^{\text{TS}}, \mathbb{Z}^{\text{PF}}\right) = 0$.

straightforward. However, in the case of our baseline model, household heterogeneity introduces one *numerical* equilibrium condition – equation (13) – whose derivatives must also be obtained numerically, which in turn can be computationally burdensome. We address this challenge by drawing on the methodology introduced by Auclert et al. (2021), and extending it to account for aggregate uncertainty.

The system of equations (16) is of high dimension, due to all the possible combinations of time and contingencies. In particular, its dimensions are $(n_E \times (T+1) \times \tau^{\max}$, where n_E represents the number of equilibrium equations in each period in the model. In our case, we have $n_E = 5$ and set $\tau^{\max} = 100$ and $T = 300$ in our basic implementation, meaning that $\mathbf{F}(\cdot)$ is a vector with around 150 thousand rows. Accordingly, the Jacobian with respect to the general equilibrium inputs would contain more than 90 million entries. However, the structure of our shock allows us to divide the equilibrium conditions in two groups, one corresponding to the TS branch, and one corresponding to the entire set of PF branches, which in turn dramatically reduces the computational burden for solving the model. We now introduce further notation required to explain our solution method, making the distinction between the TS and the PF subsections of the economy explicit.

From inputs to outputs. We now recast the representation of a heterogeneous-agent model with aggregate uncertainty of the type proposed in section 3 as a mapping from aggregate inputs X_t and X_t^τ into outputs $\mathcal{Y}_t$ and $\mathcal{Y}_t^\tau$. Each component of X_t, as well as X_t^τ, has n_x inputs, while each component of $\mathcal{Y}_t$, as well as $\mathcal{Y}_t^\tau$, displays n_y outputs. As in Section 3.1 of Auclert et al. (2021), we define $\mathbf{v}_t$ and $\mathbf{v}_t^\tau$ as the vector representation of the value function (6), respectively in the TS and in the PF contingencies, and assume the existence of functions $y(\cdot)$ and $y^{TS}(\cdot)$, functions $v(\cdot)$ and $v^{TS}(\cdot)$, and transition matrices $\Lambda(\cdot)$ and $\Lambda^{TS}(\cdot)$ such that, conditional on the initial distribution D_0, the set of outcomes $\mathcal{Y} \equiv \{\mathcal{Y}_t, \mathcal{Y}_t^\tau\}$ solve the following system of equations:[20]

[20] As explained in Auclert et al. (2021), $\mathbf{v}_t$ and $\mathbf{v}_t^\tau$ do not necessarily need to be vector representations of (6), but can also be in the form of its derivative, which allows the application of the endogenous grid method as in Carroll (2006).

$$\mathbf{v}_t = v^{TS}\left(\mathbf{v}_{t+1}, \mathbf{v}_{t+1}^{t+1}, X_t\right) \tag{17}$$

$$D_{t+1}^{t+1} = D_{t+1} = \Lambda^{TS}\left(\mathbf{v}_{t+1}, \mathbf{v}_{t+1}^{t+1}, X_t\right)' D_t \tag{18}$$

$$\mathcal{Y}_t = y^{TS}\left(\mathbf{v}_{t+1}, \mathbf{v}_{t+1}^{t+1}, X_t\right)' D_t \tag{19}$$

$$\mathbf{v}_t^{\tau} = v\left(\mathbf{v}_{t+1}^{\tau}, X_t^{\tau}\right) \tag{20}$$

$$D_{t+1}^{\tau} = \Lambda\left(\mathbf{v}_{t+1}^{\tau}, X_t^{\tau}\right)' D_t^{\tau} \tag{21}$$

$$\mathcal{Y}_t^{\tau} = y\left(\mathbf{v}_{t+1}^{\tau}, X_t^{\tau}\right)' D_t^{\tau} \tag{22}$$

Equations (20)-(22) are analogue to equations (10)-(12) in Auclert et al. (2021). However, in our case, because there is uncertainty regarding when the regime will revert the economy can follow $\tau^{\max}$ possible perfect-foresight paths, indexed by τ. Equation (20) translates future value functions and current inputs into current value functions; equation (21) in turn provides a (linear) mapping between today's and tomorrow's distributions, through the matrix $\Lambda\left(\mathbf{v}_{t+1}^{\tau} X_t^{\tau}\right)$; and equation (22) computes (aggregate) outcomes $\mathcal{Y}_t^{\tau}$ based on individual decisions $y\left(\mathbf{v}_{t+1}^{\tau}, X_t^{\tau}\right)$ aggregated using the distribution over individual states.

Equations (17)-(19), on the other hand, are unique to our setup and represent the part of the economy in which the uncertainty has yet to be resolved.[21] They explicitly take uncertainty into account: the first two arguments of the functions $v^{TS}(\cdot)$, $\Lambda^{TS}(\cdot)$, and $y^{TS}(\cdot)$ correspond to the two distinct future value functions, on the TS path and on the "t+1" PF path respectively. In addition, note that the future distribution determined by equation (18) will be the same if the economy continues in the TS branch (D_{t+1}) or if the shock reverts (D_{t+1}^{t+1}).

Example - One Asset HANK Model - In the model presented in section 3, there are five inputs, with $X_t = \{Y_t, \Pi_t, b_t, t_t, R_t\}$ (and similarly for X_t^{τ}).[22] For the output, we select aggregate savings, as this is the relevant object for market clearing. Thus $\mathcal{Y}_t = y^a(\mathbf{v}_{t+1}, \mathbf{v}_{t+1}^{t+1}, X_t)' D_t$, with y^a representing the asset policy function (and similarly for $\mathcal{Y}_t^{\tau}$). Finally, the asset market clearing is given by $b_t = \mathcal{Y}_t$ (and $b_t^{\tau} = \mathcal{Y}_t^{\tau}$).

Our goal is find the set of inputs that ensures that the system (16) holds, given the set of equations that translates inputs into outputs ((17)-(22)). In our standard HANK model,

[21] Note that, in fact, $v\left(\mathbf{v}_{t+1}^{\tau}, \mathbf{X}_t^{\tau}\right) = v^{TS}\left(\mathbf{v}_{t+1}^{\tau}, \mathbf{v}_{t+1}^{\tau}, \mathbf{X}_t^{\tau}\right)$, and there are analogue representations for equations (21) and (22).

[22] In practice, the dimension of X_t could be reduced using a directed acyclical graph (DAG) representation. See Auclert et al. (2021) for details.

$F\left(\mathbb{X}^{\mathrm{TS}}, \mathbb{X}^{\mathrm{PF}}, \mathbb{Z}^{\mathrm{TS}}\right)$ consists of $5 \times (T+1) \times \tau^{\max}$ (stacked) equilibrium conditions, given by expressions (9)-(13). The solution thus requires $5 \times \tau^{\max}$ inputs from $\mathbb{X}^{TS}$ and $5 \times \tau^{\max} \times T$ inputs $\mathbb{X}^{PF}$.

We are now ready to explain each step of our algorithm in further detail.

4.2 Solving for the Equilibrium in Perfect Foresight

Consider the situation in which the shock just reverted at time τ (i.e. we are at time $t = \tau$ and agents know the economy is under contingency τ). The state of the economy in the first period of contingency τ is characterized by the vector of inputs $X_{\tau-1}$ and the distribution D_τ^τ. We can then represent an equilibrium in a generic contingency τ by the following system of equations:

$$\mathbf{F}^{\mathrm{PF}}\left(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1}\right) = 0. \tag{23}$$

As each contingency features distinct initial conditions, the set of inputs $\mathbb{X}^\tau$ that solves (23) in each of them is different. Thus, we are faced with solving $\tau^{\max}$ *different* perfect-foresight equilibria. Therefore, our structure is distinct to the one proposed in Auclert et al. (2021), as our problem augmented by the fact that initial conditions are not necessarily the steady-state ones. To deal with this, we treat the initial state of the economy similarly to the way we treat inputs, as an argument of F^{PF}. In other words, one could say that we treat the initial conditions as "shocks" to the system (23).

We begin by solving (23) via perturbation. The methodology easily extends the solution to an *exact* (non-linear) perfect-foresight equilibrium, and we will return to it at the end of this section. The first-order approximation of (23) around the steady state reads:

$$\mathbf{F}_X^{PF} d\mathbb{X}^\tau + \mathbf{F}_D^{PF} dD_\tau^\tau + \mathbf{F}_{X_{\tau-1}}^{PF} dX_{\tau-1} = 0. \tag{24}$$

In the equation above, differentials (d) are taken relative to the steady state, i.e. $d\mathbb{X}^\tau = \mathbb{X}^\tau - \mathbb{X}_{ss}$, $dD_\tau^\tau = D_\tau^\tau - D_{ss}$, and $dX_{\tau-1} = X_{\tau-1} - X_{ss}$. $\mathbf{F}_X^{PF}$ represents the Jacobian of equilibrium conditions with respect to the entire path of inputs in contingency τ, whose dimension is $n_x \times T$, evaluated at the steady state. The term $\mathbf{F}_D^{PF} dD_\tau^\tau$ evaluates how equilibrium conditions at each period of the contingency are impacted by changes in the distribution D_τ^τ only, while the term $\mathbf{F}_{X_{\tau-1}}^{PF} dX_{\tau-1}$ evaluates the impact of pre-determined inputs $X_{\tau-1}$.

Rearranging expression (24), we obtain:

$$d\mathbb{X}^{\tau} = \mathbf{F}_X^{PF^{-1}} \left(F_D^{PF} dD_{\tau}^{\tau} + \mathbf{F}_{X_{\tau-1}}^{PF} dX_{\tau-1} \right) \tag{25}$$

The expression above is quite intuitive. It states that in order to compute equilibrium changes in inputs $d\mathbb{X}^{\tau}$, we need to understand two elements: first, how changes in *initial conditions* affect equilibrium conditions, which is represented by the expression in parenthesis; second, we compute how changes in *inputs* will affect equilibrium conditions, which is given by the Jacobian F_X^{PF}.

As at this step of our algorithm we are considering a perfect-foresight path, we directly employ the methodology in Auclert et al. (2021) to compute $\mathbf{F}_X^{PF}$. The novelty of our method involves the term in parenthesis for us it to compute the terms $\mathbf{F}_D^{PF} dD_{\tau}^{\tau}$ and $\mathbf{F}_{X_{\tau-1}}^{PF} dX_{\tau-1}$.

We compute the term $\mathbf{F}_D^{PF} dD_{\tau}^{\tau}$ numerically, making use of the interpretation provided above. In other words, we obtain the impact of a change in the distribution D (relative to the steady state) on equilibrium conditions by computing:

$$\mathbf{F}_D^{PF} dD_{\tau}^{\tau} \approx \mathbf{F}^{PF} \left(\mathbb{X}_{ss}^{PF} | D_{\tau}^{\tau}, X_{ss} \right) - \mathbf{F}^{PF} \left(\mathbb{X}_{ss}^{PF} | D_{ss} X_{ss} \right) \tag{26}$$

The expression above yields a vector consisting of $n_y \times T$ entries, each one representing the evaluation of one (out of n_y) equilibrium condition at a given period in the perfect-foresight path τ.

Note that to obtain (26), one possibility is to simulate the economy along the entire contingency τ, conditional on the initial distribution D_{τ}^{τ} and steady-state inputs. As we need to repeat this step at $\tau^{\max}$ times for each iteration of algorithm 1, this approach in practice is slow. Instead, we again follow a key insight from Auclert et al. (2021): as we treat changes in initial conditions as a one-time *shock*, we use insights from lemma 3 and proposition 1 to efficiently compute the derivative of equilibrium conditions with respect to this shock at all horizons along the PF path.

In the case of our one-asset HANK model, changes in the initial distribution of households over states will only directly affect the total supply of savings at all periods in a given contingency τ, as this is the only endogenous household decision the model features. Let F_t^{τ} be one entry of $\mathbf{F}^{PF}$ representing the asset market clearing condition at time t, contingency τ:

$$\begin{aligned} F_t^\tau(\mathbb{X}^\tau, D_\tau^\tau) &= y^a\left(\mathbf{v}_{t+1}^\tau, \mathbf{X}_t^\tau\right)' D_\tau^\tau - b_t^\tau \\ &\equiv \mathcal{Y}_t^\tau - b_t^\tau \end{aligned}$$

The term b_t^τ equation above is independent of the heterogeneous-agent block. The function y^a represents the individual policies for asset holdings. Our difficulty thus relies on computing derivatives of the first term. For that, we use:

$$d\mathcal{Y}_t^\tau = y_{ss}^{a\,\prime}(\Lambda_{ss}')^{t-\tau} dD_\tau^\tau \tag{27}$$

The expression above is similar to the first column of equation (26) in Auclert et al. (2021), and computes the asset demand for when households display their steady-state policies, but the initial distribution has changed to D_τ^τ.[23] The intuition for expression (27) is the following: even though at date zero, the idiosyncratic distribution of households over states is different than the steady-state one, moving forward households maintain policies, and thus the distribution D converges back to the D_{ss} over time, the convergence being dictated by the transition matrix Λ_{ss}.

The key advantage of exploiting equation (27) is that for each t, the linear transformation $y_{ss}^{a\,\prime}(\Lambda_{ss}')^{t-1}$ can be *pre-computed* and stored, and therefore recycled at each iteration of step 1 in algorithm 1 (see Lemma 3 in Auclert et al. (2021)). This way, by computing heterogeneous-agents Jacobians with the use of (27), we can efficiently compute expression (26).

Because of our assumed fiscal rule, our baseline model does not feature pre-determined variables other than the distribution over idiosyncratic states, and, thus, we relegate the discussion of the computation of the last term in (26) to the appendix.

In the numerical implementation of the model presented, we discretize D using n_a points for assets and n_z points for labor productivity, so D is a $n_a \cdot n_z \times 1$ vector. To compute the equilibrium in each contingency, we take the pre-determined (guessed) distribution at each τ and compute (26) as described above. This step is not computationally demanding, as we pre-compute the T matrices $y_{ss}^{a\,\prime}(\Lambda_{ss}')^{t-\tau}$ in advance.

[23] Specifically, we are considering the first column of the “fake news” matrix.

Occasionally Binding Constraints - The Zero Lower Bound. Our solution methodology can also accommodate the presence of non-linearities on the behavior of the (macro) economy. In our applications, the Central Bank is constrained by the ZLB on nominal interest rates.

To deal with occasionally binding constraints we follow the approach of Guerrieri and Iacoviello (2015): in each branch, we first compute dX^{τ} without imposing the bound. This gives us the shadow rates $SR^{\tau} \equiv \{SR_t^{\tau}\}_{t=\tau}^{T+\tau-1}$ - the nominal rates the Central Bank would select if it were unconstrained. Finally, we then reset $R_t^{\tau} = 1$ at each period in which $SR_t \leq 1$ and readjust the Jacobian to account for the fact that the central bank is constrained in those periods. We repeat the approach until the set of periods in which the ZLB binds is stable.[24]

We are now ready to detail each step of step 2 of Algorithm 1.

Algorithm 2 ***Occasionally Binding Constraints on PF Contingencies.*** *Given* $\{D_{\tau}^{\tau}\}_{\tau=1}^{\tau^{\max}}$ *and* $\{\mathbb{X}^{\tau}\}_{\tau=0}^{\tau^{\max}-1}$, *initialize the set of periods in which the ZLB binds* $o^n = \emptyset$, $n = 0$.

1. *Compute* $d\mathbb{X}^{\tau}$ *following expression 25 in each contingency.*
2. *Compute the shadow rates* SR_t^{τ} *for each* t *and* τ
3. *Compute* $o^{n+1} = \{t, \tau\}$ *such that* $SR_t^{\tau} \leq 1$. *In the set of model equilibrium equations, substitute the Taylor Rule for all* $\{\tau, t\} \in o$ *for* $R_t^{\tau} = 1$. *Modify the Jacobian* F_X^{PF} *accordingly.*
4. *If* $o \neq o^{n+1}$, *return to 1.*
5. *Proceed to step 3 in Algorithm 1.*

Note that, conditional on the set o^n, in the algorithm above the solution to each PF contingency is obtained via first-order perturbation. We explain how we find their *exact* solution below. Finally, also note that, given initial conditions D_{τ}^{τ} and $X_{\tau-1}$, the steps 1-5 above are independent across contingencies, and thus fit to parallelization.

4.2.1 *Exact* Equilibrium

Equation (25) computes the equilibrium inputs in each contingency using a first-order perturbation, which approximates the solution to the system of equations (23).[25] Because the asset

[24] See Ascari and Mavroeidis (2022) and Holden (2023) for a discussion on existence and uniqueness of equilibrium at the ZLB under perfect foresight.

[25] The other equilibrium conditions will be exactly satisfied, as we use their linearized version.

demand function is non-linear on X, there might be an approximation inaccuracy. This is particularly likely to happen if the initial distribution D_τ^τ is too distant from the steady-state one, or with strong non-linearities in the behavior of the economy, as in the case of the zero lower bound. We discuss the accuracy of first-order solutions in further the detail at the end of this section, when we discuss the implementation of algorithm 1.

It is straightforward to test if our approximated solution, given by $\mathbb{X}^\tau \equiv \mathbb{X}_{ss}^{PF} + d\mathbb{X}^\tau$, is accurate, i.e. approximates well the solution to the system (23). We can forward-simulate the economy along each contingency τ and evaluate the whole set of equilibrium conditions. This, however, can be a burdensome step, as it involves the computation of several transition matrices Λ along each ot the $\tau^{\max}$ PF paths. On the other hand, the procedure is also fit to parallelization, as, for a given $\mathbb{X}^\tau$, $\tau \in \{1, 2, ..., \tau^{\max}\}$, the branches are completely independent.

Moreover, we can exploit the Jacobian $\mathbb{F}_X^{PF}$ to devise a quasi-Newton method and solve for the *exact* equilibrium along each perfect-foresight branch. This requires repeated iterations of algorithm 2. In fact, algorithm 2 consists of the first step of a quasi-Newton method where further iterations make use of the Jacobian F_X^{PF}. The complete algorithm for finding the exact equilibrium (with the possibility of a binding ZLB) along each PF branch is the following.

Algorithm 3 ***Perfect Foresight Contingencies - Exact Equilibrium with ZLB*** *Given* $\{D_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$ *and* $\{\mathbb{X}^\tau\}_{\tau=0}^{\tau^{\max}-1}$*, initialize the set of periods in which the ZLB binds* $o^n = \emptyset$, $n = 0$.

1. *Perform steps 1-5 in Algorithm 2, obtaining* $\mathbb{X}^{\tau,0} = \mathbb{X}_{ss} + d\mathbb{X}^\tau$

2. *Compute* $\mathbf{F}^{PF}(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1})$ *by forward-simulating the economy along all contingencies.*

3. *If* $||\mathbf{F}^{PF}(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1})|| \leq \epsilon$ *for a given* $\epsilon > 0$*, conditional on* o^n*, the exact equilibrium is found (up to the tolerance* ϵ*). If not, update the endogenous variables in each contingency according to the formula:*

$$\mathbb{X}^{\tau,m+1} = \mathbb{X}^{\tau,m} - \mathbf{F}_X^{PF^{-1}} \mathbf{F}^{PF}(\mathbb{X}^\tau | D_\tau^\tau, X_{\tau-1})$$

   and return to step 2.

4. *Using the resulting* $\mathbb{X}^\tau$*, perform steps 2-3 in algorithm 2.*

5. *If* $o^n \neq o^{n+1}$*, return to step 1. Else,* $\mathbb{X}^\tau$ *represents the exact equilibrium inputs for contingency* τ*, given on pre-set initial conditions.*

We now proceed to step 3 in Algorithm 1.

4.3 Solving for the Equilibrium with Aggregate Uncertainty

In step 3 of algorithm 1, the perfect-foresight paths the economy can pursue after leaving the two-state branch are taken as given. In fact, due to the recursive nature of consumer's problem, solving the equilibrium in the TS branch only requires knowledge of the the value functions and the set of inputs in the initial period of each perfect-foresight contingency. We denote the (stacked) vector of value functions $(\{V_\tau^\tau\}_{\tau=1}^{\tau^{\max}})$ in the initial period of each contingency by $\mathbb{V}_1^{PF}$ and the analogue stacked vector of inputs $(\{X_\tau^\tau\}_{\tau=1}^{\tau^{\max}})$ by $\mathbb{X}_1^{PF}$. The equilibrium in the TS branch is characterized by:

$$\mathbf{F}^{\mathrm{TS}}\left(\mathbb{X}^{TS}, \mathbb{Z}^{TS} \middle| \mathbb{X}_1^{PF}, \mathbb{V}_1^{PF}\right) = 0, \tag{28}$$

given that the initial conditions $D_0 = D_{ss}$ and $X_0 = \mathbf{X}_{ss}$.

In the TS branch, the model outputs are characterized by equations (17)-(19). This step of the algorithm takes future values V_{t+1}^{t+1} as given and solves the system of equations for X_t. As before, we first describe how we solve find the equilibrium in TS by perturbation. To a first order:

$$\mathbf{F}_X^{TS} d\mathbb{X}^{TS} + \mathbf{F}_Z^{TS} d\mathbb{Z}^{TS} + \mathbf{F}_{\mathbb{X}}^{TS} d\mathbb{X}_1^{PF} + \mathbf{F}_{\mathbb{V}}^{TS} d\mathbb{V}_1^{PF} = 0$$

And rearranging:

$$d\mathbb{X}^{TS} = (\mathbb{F}_X^{TS})^{-1}\left(\mathbf{F}_{\mathbb{Z}}^{TS} d\mathbb{Z} + \mathbf{F}_{\mathbb{X}}^{TS} d\mathbb{X}_1^{PF} + \mathbf{F}_{\mathbf{V}}^{TS} d\mathbb{V}_1^{PF}\right) \tag{29}$$

In the TS branch, we treat *future* conditions the same way we treat *initial* conditions in equation (24): as shocks, i.e. we take them as given. In fact, the last two terms on the right-hand-side of (29) are somewhat analogue to the last two terms of equation (24), in the sense that they are conditioned upon.

The last term in equation (29) represents the impact of changes in households' future value functions - at the initial period of the PF branches - on equilibrium conditions along the TS branch. Once again, we compute it making use of the following:

$$\mathbf{F}_{\mathbb{V}}^{TS} d\mathbb{V}_1^{PF} \approx \mathbf{F}^{TS}\left(\mathbb{X}_{ss}, \mathbb{Z}_{ss} | \mathbb{X}_{ss}, \mathbb{V}_1^{PF}\right) - \mathbf{F}^{TS}\left(\mathbb{X}_{ss}^{TS}, \mathbb{Z}_{ss} | \mathbb{X}_{ss}^{TS}, \mathbb{V}_{ss}\right),$$

where $\mathbb{Z}_{ss}$ represent a stacked vector of shocks at their steady-state values. The computation of the expression above is done by solving the households' problem and forward-simulating the economy along the two-state branch in response *only* to the changes in $\mathbb{V}_1^{TS}$, with all other inputs at their steady-state values. At each iteration of algorithm 1, it has to be done once.

Changes in *future* inputs (X_τ^τ) on *current* household policies can only have an effect through changes in households' future value functions (see equations (17) and (20)). More generally, inputs can impact current equilibrium conditions via forward-looking terms in analytical equilibrium conditions, which is captured by $F_{\mathbb{X}}^{TS} d\mathbb{X}_1^{PF}$. This term can, therefore, be computed analytically.

In the case of our baseline model, the only analytical equilibrium equation with a forward-looking term is the New-Keynesian Phillips Curve (equation (9)). Solving explicitly the expectations term in this expression along the TS branch yields:

$$\begin{aligned}\left(\Pi_t - \overline{\Pi}\right)\Pi_t = \beta_t \mu \left(\frac{Y_{t+1}}{Y_t}\right)^{1-\sigma} \left(\Pi_{t+1} - \overline{\Pi}\right)\Pi_{t+1} + \\ + \beta_t (1-\mu) \left(\frac{Y_{t+1}^{t+1}}{Y_t}\right)^{1-\sigma} \left(\Pi_{t+1}^{t+1} - \overline{\Pi}\right)\Pi_{t+1}^{t+1} + \kappa\left[Y_t^{\omega+\sigma} - 1\right]\end{aligned}$$

We can then use the expression above to compute how changes in the initial period of each PF branch (in this case Π_{t+1}^{t+1} and Y_{t+1}^{t+1}) impact equilibrium conditions on the TS branch.

Returning to equation (29), the term $\mathbb{F}_{\mathbb{Z}}^{TS} d\mathbb{Z}^{TS}$ corresponds to the impact of shocks, conditional on inputs being at their steady-state values. Computing this term is done in two steps. For the heterogeneous agent block, a shock is treated as an input, i.e. we apply the same methodology used to compute $\mathbb{F}_X^{TS}$. In a second, we analytically derive the Jacobian of aggregate equilibrium conditions with respect to shocks.[26]

One of the innovations of our methodology is in computing the term $\mathbb{F}_X^{TS}$. For the equilibrium conditions with analytical representation, the corresponding Jacobian entries can also be computed analytically.[27] Our contribution instead concerns adapting the method by Auclert

[26] In the baseline version of the model, the second step is skipped since shock Z is summarized by the discount factor shock to households and is assumed to only affect the heterogeneous agent block.

[27] This procedure is essentially different from Eggertsson et al. (2021), where the expectation is explicitly solved

et al. (2021) to compute heterogeneous-agents Jacobians under aggregate uncertainty.

We only require a simple adaptation of first step of the "Fake News Algorithm". Following Auclert et al. (2021), this step first requires simulating the response of households to an announced shock s periods ahead using a single backward iteration. As with their case, we also only need to simulate the economy once, but here households do take the aggregate uncertainty in account. In particular, there is a probability $1-\mu$ that the economy will leave the TS branch at any point in time, and react accordingly.[28] The remaining steps of the "Fake News Algorithm" are unchanged.

Intuitively, aggregate uncertainty affects the reaction of households to news regarding changes in future inputs. In particular, because households attribute a probability $\mu^s < 1$ that a node in the TS branch s periods ahead will be reached, they under-react to future news, relative to the case in which $\mu = 1$.

Figure 6 shows how uncertainty affect the heterogeneous-agents Jacobians. It plots the response of aggregate savings to changes in output, i.e. a partial equilibrium "shock" in Y_s, at different horizons s, for different degrees of uncertainty μ. In particular, in line plots the derivatives $\left\{\frac{d\mathcal{Y}_t}{dY_s}\right\}_{t=0}^{\tau^{\max}-1}$, with $\mathcal{Y}_t$ representing aggregate savings. Recall that changes in output Y_s have a direct impact on individual labor income, as household i's gross earnings is given by $z_i Y_s$. Because contemporaneous change in input are certain, different values for the uncertainty parameter μ do not have a different impact on the change in households savings decisions with respect to changes in Y_0.

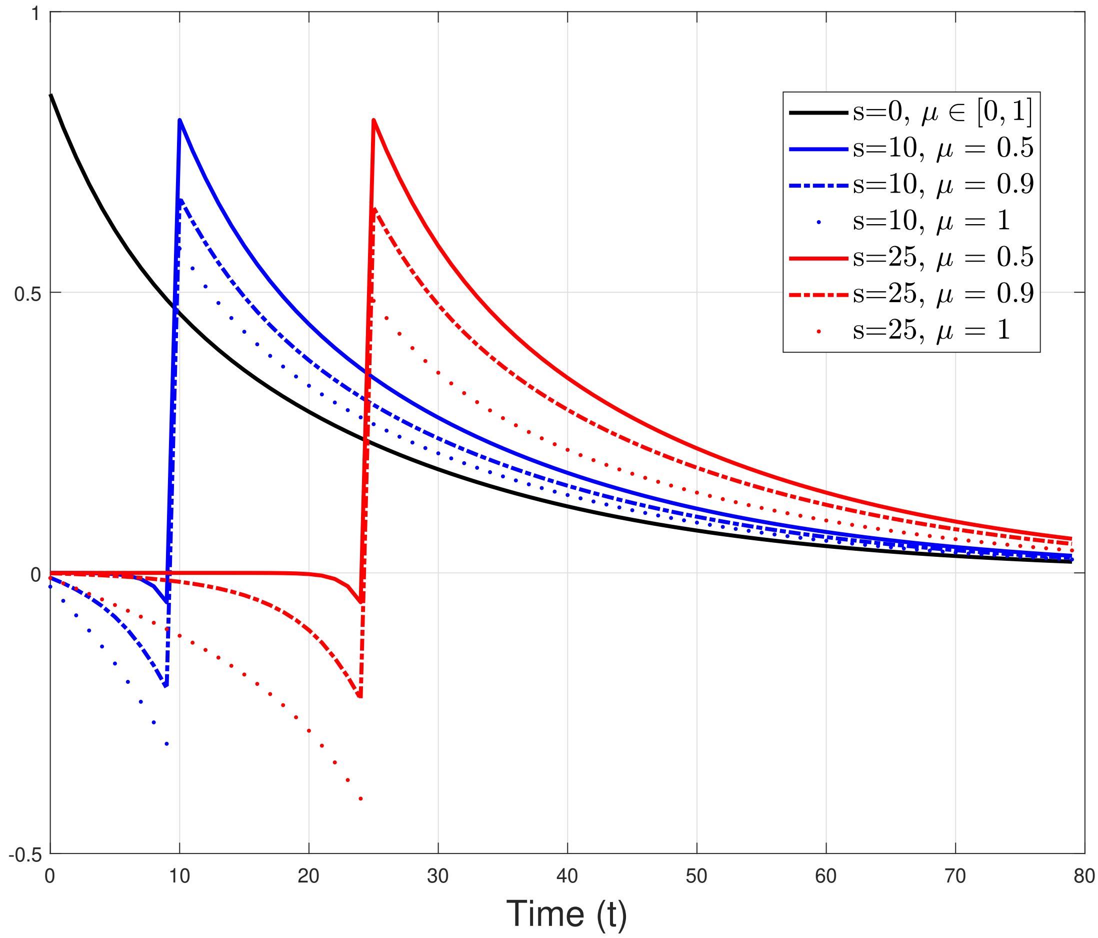
At horizons $s > 0$, though, the uncertainty matters. Recall that agents are told that there is a state of the world in which GDP is higher at some future point s, so in the times leading to such period, they start consuming part of this future income by tapping on their savings stock. The lower μ is, the weaker is the reduction in savings in anticipation of changes in output, as households attribute low probabilities to that event. The anticipation is particularly muted for distant horizons. This can be seen, for instance, in the solid red line: there is essentially no reaction to news of a change in output happening 25 periods in the future, because the perceived probability of this event actually happening is negligible. On the contrary, in the case of $\mu = 1$ (dotted line), households immediately react to the certain expectation of a change in output happening even 25 periods ahead.

by properly weighing transition matrices.

[28]The exception is $\tau^{\max}$, in which the economy leaves the TS branch with certainty.

As opposed to a reduced anticipatory response to future shocks, when the shock materializes the impact on savings is stronger with aggregate uncertainty, relative to the case when $\mu = 1$. This can be seen by comparing the solid and dashed blue and red lines with their dotted counterparts. The intuition is that, when $\mu = 1$, households front-load the consumption a relatively large portion of the expected income windfall. Instead, with uncertainty, consumption front-loading is relatively muted, and a relatively larger portion of the windfall is consumed after it materializes.

Figure 6: Asset Market Clearing Jacobian $\left(\frac{d\mathcal{Y}_t}{dY_s}\right)_{t=0}^{\tau^{\max}-1}$ for Distinct μ's



Zero Lower Bound and Exact Equilibria. Along the TS branch, we deal with occasionally binding ZLB in exactly the same way as in the PF branches, by following Guerrieri and Iacoviello (2015) (see algorithm 2). In addition, to compute the exact equilibrium along the TS branch, the steps are analogue to algorithm 3.

We now proceed to further details regarding the implementation of our algorithm.

4.4 Results - Implementation Details

To solve the household problem in the model described in section 3, we discretize the asset grid in n_a points and the income grid in n_z points, and employ the endogenous grid method proposed by Carroll (2006).

In table 2 we show the running times for each distinct specification, together with the maximum deviation of equilibrium condition (16) in each distinct setup. The benchmark model features $n_z = 3$ and $n_a = 200$. In addition, we include a case with $n_z = 15$ grid points and $n_a = 500$. In this specification, we impose that earnings follow an AR1 process whose innovation is drawn from a mix of normal distributions, and calibrate the parameters as in Mendicino, Nord and Peruffo (2021), matching high-order moments of the distribution of earnings changes.[29] We keep the aggregate shock structure the same as in the benchmark. The tolerance within successive iterations of algorithm 1 is set to 10^{-9}, the steady-state general equilibrium tolerance is set to 10^{-12}, the tolerance for computing the *exact* equilibria is set to 10^{-8}, with the max norm. Codes are written in Matlab and were ran on an ASUS laptop with 1.80Ghz processor, 16GB RAM, and 8 cores.

Table 2: Running Times

Specification	**Benchmark**		**MNP**	
Step	Time	Max. Err.	Time	Max. Err.
Steady State	0.7	-	6	-
All Jacobians	4	-	179	-
Algorithm 1 - First-Order	20	0.5%	144	0.5%
Algorithm 1 - Exact only on TS	26	0.008%	216	0.002%
Algorithm 1 - Exact Equilibrium	116	0.00000006%	7735	0.00000002%

Notes: Times are given in seconds. "Benchmark" refers to the model calibrated as in section 3, with $n_z = 3$ and $n_a = 200$, while "MNP" stands for the model calibrated as in Mendicino, Nord and Peruffo (2021), with $n_z = 15$ and $n_a = 500$. The row "Algorithm 1 - First-Order" refers to the solution of both PF and TS branches via first-order perturbation. The row "Algorithm 1 - Exact only on TS" refers to the solution of PF paths via perturbation but the exact equilibrium computed in the TS branch. The row "Algorithm 1 - Exact Equilibrium" solves for the exact equilibrium in the economy. Maximum errors correspond the maximum absolute value of the asset market clearing equilibrium condition, given as a percentage of steady-state total asset holdings.

[29] Specifically, we target the cross-sectional variance of log annual earnings, the standard deviation, the skewness and kurtosis of log annual earnings changes, and the ratio of the 90th to the 10th percentile of log changes.

From table 2, we see that under all scenarios the maximum errors are small, even with the first-order solution; for the other two cases, errors are negligible. In practice, the main results do not change across the three scenarios: for instance, the initial impact of the shock on output equals -8.02% in the first-order approximation and -8.10% in the other two cases.

Finally, it is worth highlighting that the low running times allow us to compute counterfactual simulation multiple times, which let us, for instance, consider distinct policy counterfactuals. We will exploit this advantage of our solution methodology further in sections 5 and 6.

5 Aggregate Uncertainty at the ZLB

Our main economic results from the simple model carry over to the more quantitative HANK structure: aggregate uncertainty interacts with the ZLB, and this interaction is amplified with idiosyncratic risk. In this section, we perform three experiments to illustrate those results by comparing effects on impact and discounted impulse response functions for inflation and output, $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\Pi_t - \overline{\Pi}\right)$ and $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(Y_t - \overline{Y}\right)$. First, we use the quantitative model described in section 3 and compare the effects of a stochastic shock and its deterministic counterpart in the absence of the ZLB. The shocks are calibrated to match the same output and inflation responses (on impact) as in the Great Recession, in the model with aggregate uncertainty and the ZLB. This experiment confirms that *without* the ZLB, the effects of the stochastic and deterministic shocks are identical. Second, we repeat the experiment with the imposition of a lower bound, i.e. $\underline{R} = 1$. This experiment quantifies the interaction between aggregate uncertainty and the ZLB. Third, after constructing a RANK economy whose demand shocks imply observational equivalence with the HANK model *without* lower bound, we repeat the experiment of introducing the ZLB. This analysis quantifies the role of idiosyncratic risk in determining the interaction between aggregate uncertainty and the ZLB.

5.1 IRFs when the Central Bank is not constrained

The first exercise concerns the validation of the certainty equivalence under our specified shock structure. We consider our HANK economy subject to the stochastic shock, as described by equation (15), and compare it against the same economy subject to a deterministic shock. In the second case, agents know with certainty the whole sequence of demand shocks. To make them comparable, we set the sequence of demand shocks in the deterministic case to be equal to the unconditional expectation of the stochastic case. Formally, we set $\beta_t^{DET} = \mu^t \beta_L + \left(1 - \mu^t\right)\beta$.[30]

Figure 7 shows the effects on output, inflation, and nominal rates, of the stochastic and deterministic shocks, plotting contingencies (blue solid lines), the IRF of the stochastic shock (red dotted line), and the IRF of the deterministic shock (black solid line). Notably, the impulse responses under the two shocks are essentially identical, with a 3.75% recession on impact and inflation at 0.8% (below the 2% target). Those effects are reached with nominal interest rates actually becoming negative, as shown in the third panel. Since the average responses are the same, their discounted sums are also identical, as we report in the first column of figure 11, which plots the discounted IRFs for output (left) and inflation (right) relative to the corresponding ones under a deterministic shock (blue bar, first column).

This result confirms that certainty equivalence holds in our model, despite the non-linearities at the individual level. In other words, the impact of a stochastic shock can be well approximated by simulating the response of the economy to its expected value. This equivalence result partly relies on the fact that the response of aggregate variables in the model are approximately linear. In the next exercise, we show that the introduction of the ZLB under the same shocks breaks the equivalence result.

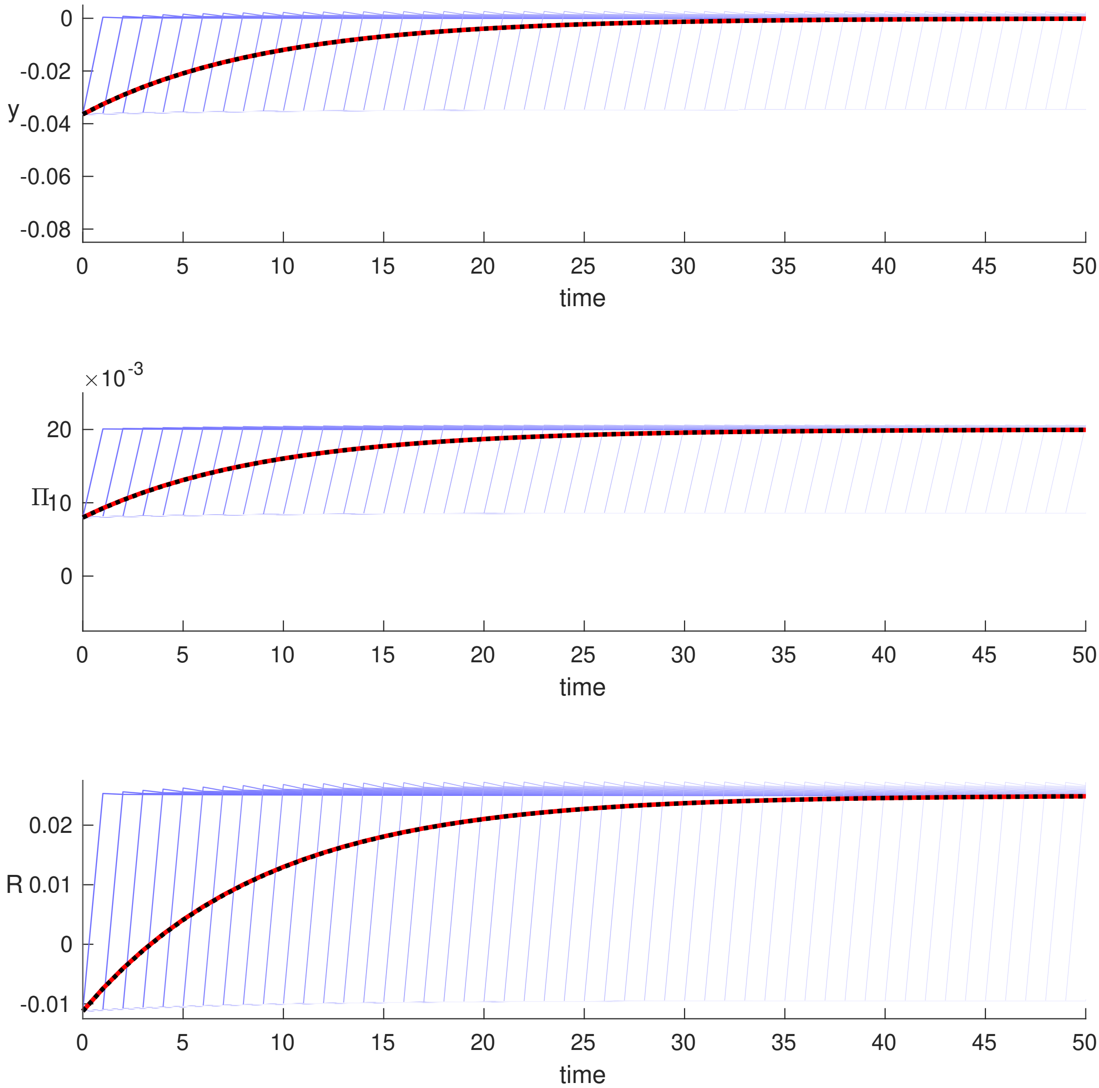
5.2 Introducing the ZLB

To assess the interaction between aggregate uncertainty and the ZLB, we repeat the previous experiment with the only difference that the lower bound to nominal rates is now set to 0 ($\underline{R} = 1$). As shown in the bottom panel in figure 7, nominal rates were to be negative in reaction to the shocks. Introducing the ZLB will make nominal interest rates higher than otherwise, further depressing aggregate demand, and eventually amplifying the recession. The economic mechanisms at work are the same as in McKay, Nakamura and Steinsson (2016).

Figure 8 plots the effects on output, inflation, and nominal rates, of the stochastic and deterministic shocks, plotting contingencies (blue solid lines), the IRF of the stochastic shock (red solid line), and the IRF of the deterministic shock (black dotted line). It shows our first result: introducing the ZLB generates a dichotomy between the impulse responses under the

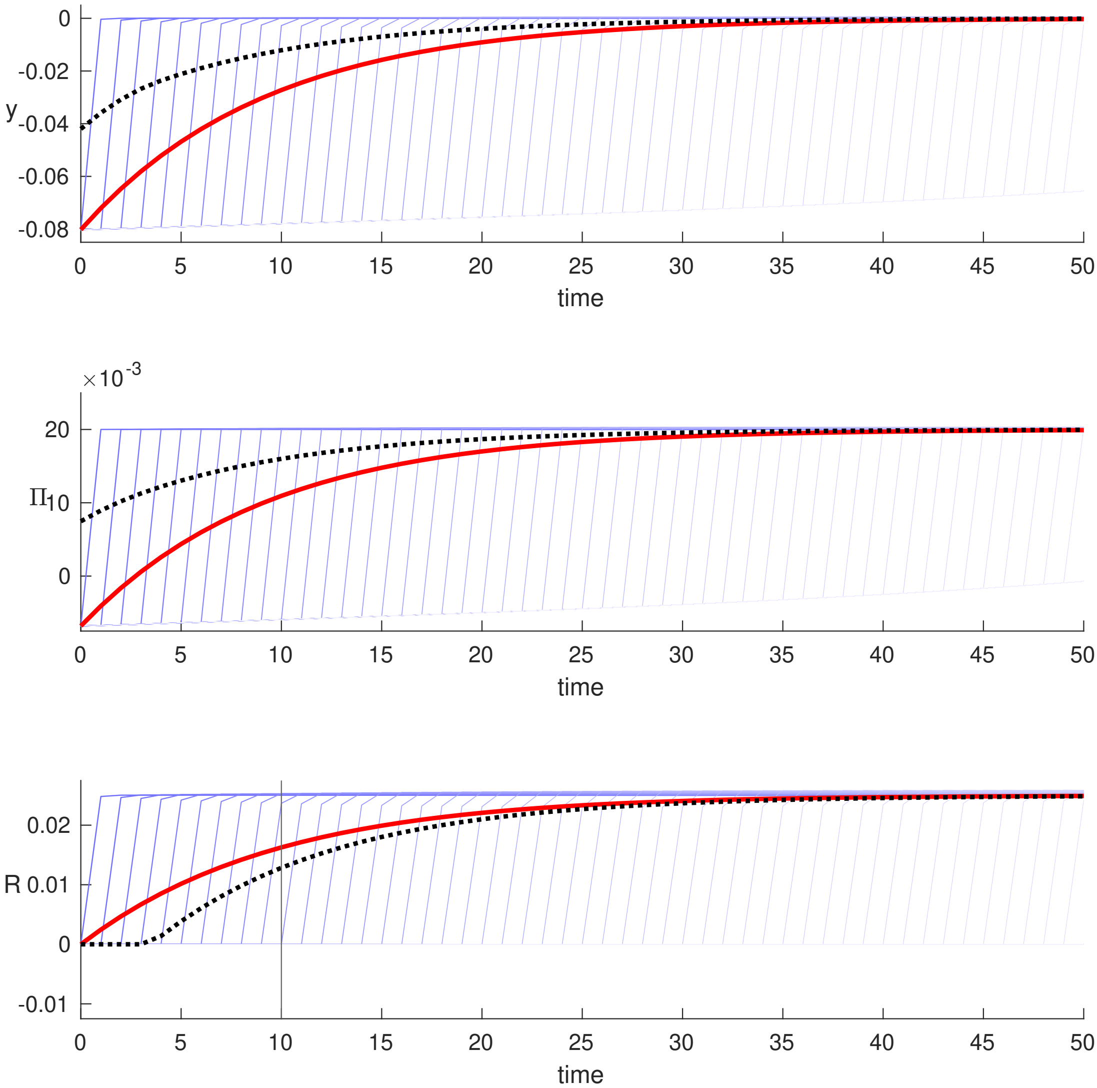
[30]This step is slightly different from the one described in section 2, since we are not targeting the same effects here. It can be interpreted as a first-order approximation to what we do in the simple model, but nevertheless we consider this to be a rather conservative assumption, since it implies a larger effect under the deterministic shock, as can be seen in the top-right panel of figure 2.

Figure 7: IRF and Contingencies - Outside the ZLB



Notes: The figure reports the effect of a stochastic demand shock as defined in equation (15) (each blue line corresponds to one individual contingency, with thickness proportional to its unconditional probability, the red solid line is the impulse response function obtained as a weighted average across all contingencies) and its deterministic counterpart (black dotted line), in our baseline HANK model ignoring the ZLB (i.e. imposing $\underline{R} = -\infty$). The first panel reports the effects on output, in deviation from steady state. The second and third panels correspond to annualized inflation and nominal interest rate levels. The x-axis in all panels measures time in quarters.

Figure 8: IRF and Contingencies - With the ZLB



Notes: The figure reports the effect of a stochastic demand shock as defined in equation (15) (each blue line corresponds to one individual contingency, with thickness proportional to its unconditional probability, the red solid line is the impulse response function obtained as a weighted average across all contingencies) and its deterministic counterpart (black dotted line), in our baseline HANK model with the ZLB (i.e. imposing $\underline{R} = 1$). The first panel reports the effects on output, in deviation from steady state. The second and third panels correspond to annualized inflation and nominal interest rate levels. The black vertical solid line reports the expected duration of the ZLB under the stochastic shock. The x-axis in all panels measures time in quarters.

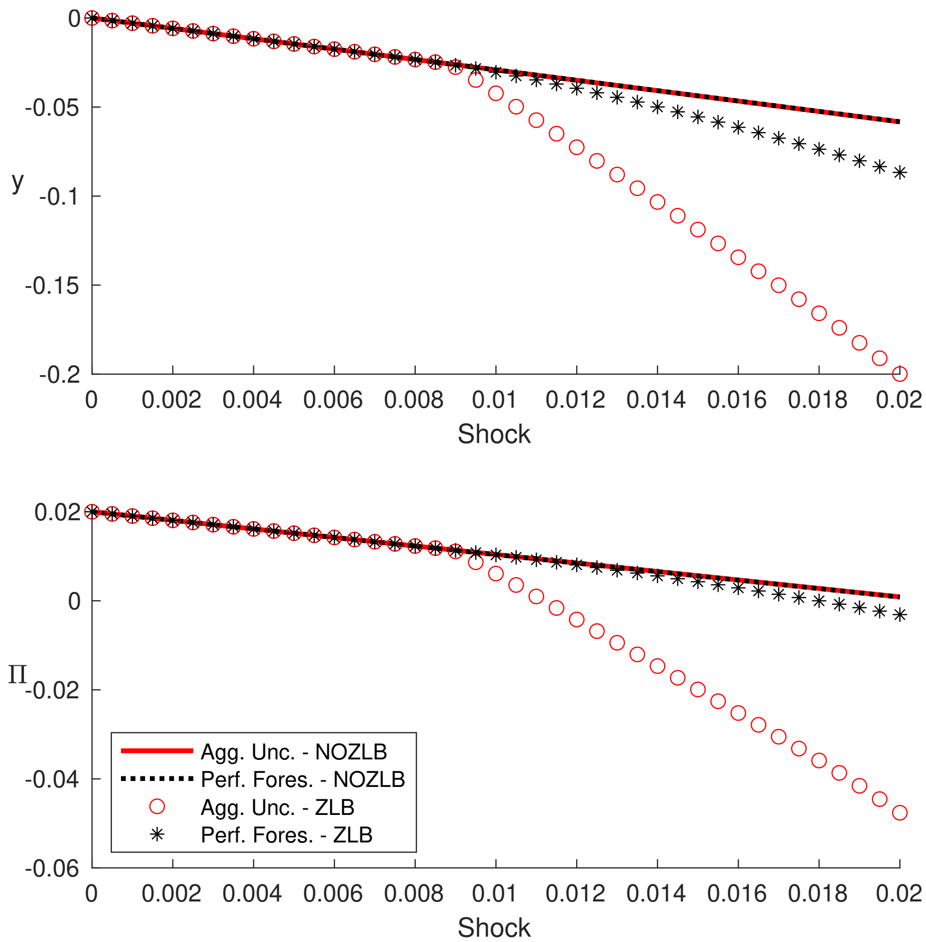
two shocks. Chiefly, the recession on impact under the stochastic shock (8%) is about double the size of that under the deterministic shock (4%). Similarly, while the price dynamics implies an inflation level of 0.7% under the deterministic shock, the model predicts a mild deflation on impact under the stochastic shock.

The second column of each panel in figure 11 plots the corresponding discounted IRFs for output (left) and inflation (right), under the deterministic (blue bar) and stochastic (red bar) shock, relative to the corresponding discounted IRF under the deterministic shock in the model without the ZLB. The introduction of the ZLB increases the expected loss in terms of output by 4.5% in the deterministic case and by 125% in the stochastic case (for inflation the effects are 1.9% and 127%). Those results establish that the ZLB can break certainty equivalence and, crucially, confirm that there is a strong economic interaction with aggregate uncertainty.

A smaller point must be made about the plots on nominal interest rates. If one is interested in understanding the expected duration of the liquidity trap under a deterministic scenario, it suffices to focus on the time at which the black dotted line for nominal rates lifts-off from its lower bound. The same cannot be said under the stochastic shock. By merely looking at the IRF for nominal rates, one would be easily misled to believe that with the stochastic shock, the expected duration of the liquidity trap would be 0. Instead, if one computed the actual expected duration, one would find that it is at least twice as large when compared to the deterministic case. This can be seen in the bottom panel of Figure 8, where the vertical line corresponds to the expected time of lift-off for policy rates under the stochastic case and is about twice as large when compared to the deterministic case.

An alternative way to assess the strength of the economic interaction between aggregate uncertainty and the ZLB is by comparing the effects of differently sized shocks. We consider the same exercises as done so far, but for varying shock sizes. This is simply achieved by lowering β_L towards β for the stochastic scenario (and adjusting the deterministic shock accordingly). We run the simulations with and without the ZLB, as in in section 5.1 in Fernández-Villaverde et al. (2023). Figure 9 plots effects on output (top) and inflation (bottom) on impact as a function of the shock size $\beta_L - \beta$. As the shock size increases, the effects on impact increase (generating a larger recession and imposing downward pressure on inflation) in the four cases considered. The effect is linear in the shock size, but most importantly identical in the stochastic (red solid line) and deterministic case (black dotted line) when we ignore the ZLB. On the other hand, once the ZLB is taken into account, the linearity breaks once the economy enters the liquidity trap. Consistent with the example in the simple model in section 2, there are shock values that trigger the liquidity trap under the stochastic shock (red circles) but not under the deterministic shock (black stars). In other words, the threshold shock value such that the central bank becomes constrained is lower with aggregate uncertainty. Furthermore, the marginal effects are much larger under the stochastic shock, as can be seen by the steeper slope of the red circles when compared to the black stars. This is a compound effect of the more complex uncertainty structure of our quantitative model as opposed to the simple model.

Figure 9: Effects on Output and Inflation on Impact



Notes: The figure plots the effects on output (in deviation from steady state) and inflation (in annualized levels) of a demand shock as described in equation (15). The shock size, $\beta_L - \beta$, varies on the x-axis. The red solid (black dotted) line corresponds to the HANK model, with the stochastic (deterministic) shock and without the ZLB. The red circles (black stars) correspond to the HANK model, with the stochastic (deterministic) shock and with the ZLB.

5.3 The Role of Heterogeneous Agents

What is the role of household heterogeneity in the results above? We now quantify the role of heterogeneity by studying a RANK model and replicating the experiments done in the previous two subsections. To do that, we substitute the model block that entails household heterogeneity and substitute it with a standard intertemporal optimization condition, i.e. the consumption Euler equation:

$$C_t^{-\sigma} = \beta_t^{RANK} R_t \mathbb{E}_t \frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}}. \tag{30}$$

To make models comparable, we calibrate the discount factors β^{RANK} under the stochastic case so that they perfectly match the effects on output and inflation in the corresponding contingencies of the HANK model. As a deterministic shock, we consider the unconditional expectations of the stochastic shocks, as done in the previous exercise.[31]

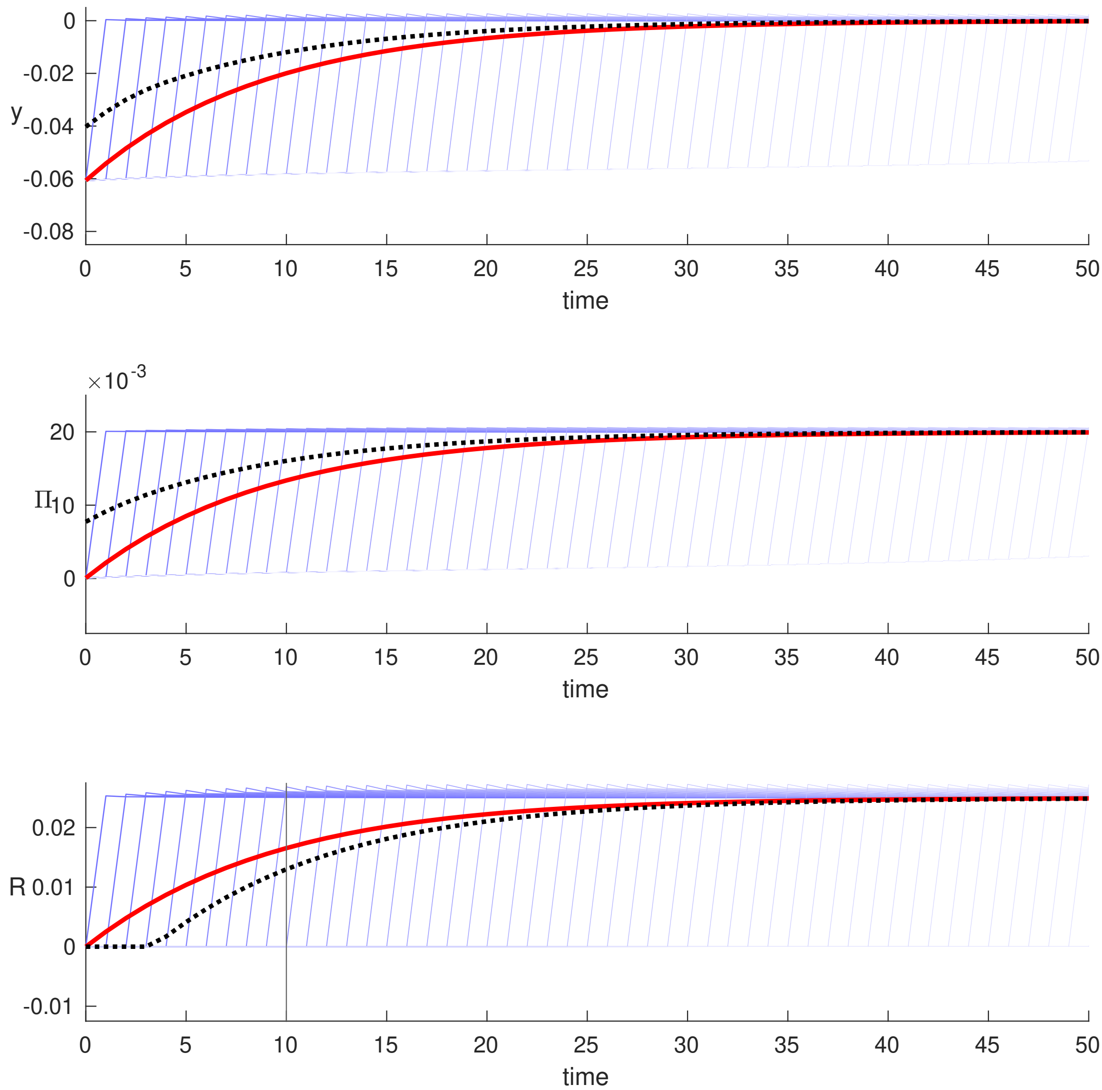
When we ignore the ZLB, the resulting simulations in the representative agent model are identical to the corresponding ones in the HANK model, both under the stochastic shock (by construction) and deterministic shock (due to certainty equivalence). Once we introduce the ZLB, the effects that materialize on output and inflation are similar: a dichotomy emerges between the impulse responses under the two shocks. Similar to what happens in the HANK model, the recession on impact under the stochastic shock (6%) is larger than that under the deterministic shock (4%), while the price dynamics implies an inflation level of 0.7% under the deterministic shock, the model predicts 0 inflation on impact under the stochastic shock.

The third column of each panel in figure 11 plots discounted IRFs for output (left) and inflation (right) of the RANK model with ZLB, under the deterministic (blue bar) and stochastic (red bar) shock, relative to the corresponding discounted IRF under the deterministic shock in the HANK model without the ZLB.[32] The introduction of the ZLB in the RANK model increases the expected loss in terms of output by 2.3% in the deterministic case and by 66.7% in the stochastic case (for inflation the effects are 0.45% and 67.3%). However, the effects are significantly smaller than those in the HANK economy (about halved), meaning that the

[31] The resulting shock process is slightly different from a purely Two-states Markov process, in that the discount factor levels now take more than just values across all contingencies and times. The shock structure does retain the uncertainty structure with the same fixed probability of reversal μ. Once the shock is over, this process can be rationalized as a deterministic sequence.

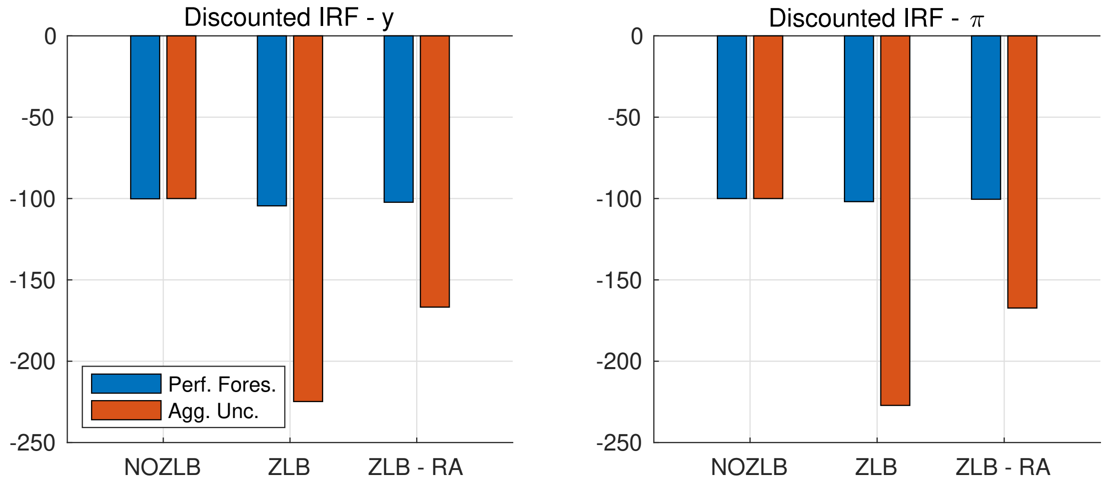
[32] We omit the bars that correspond to the RANK model without ZLB as they are identical to the ones in the first column.

Figure 10: IRF and Contingencies - With the ZLB



Notes: The figure reports the effect of a stochastic demand shock calibrated as described in the text (each blue line corresponds to one individual contingency and its thickness is proportional to its unconditional probability, the red solid line is the impulse response function obtained as a weighted average across all contingencies) and its deterministic counterpart (black dotted line), in a standard RANK model with the ZLB (i.e. imposing $\underline{R} = 1$). The first panel reports the effects on output, in deviation from steady state. The second and third panels correspond to annualized inflation and nominal interest rate levels. The black vertical solid line reports the expected duration of the ZLB under the stochastic shock. The x-axis in all panels measures time in quarters.

Figure 11: Discounted IRF - Output and Inflation



Notes: The figure reports the implied discounted impulse response functions for output (left panel) and inflation (right panel) under the HANK model without the ZLB (first column), the HANK model with the ZLB (second column), and the RANK model with the ZLB (third column). Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model without the ZLB, under the deterministic shock (left most blue bar).

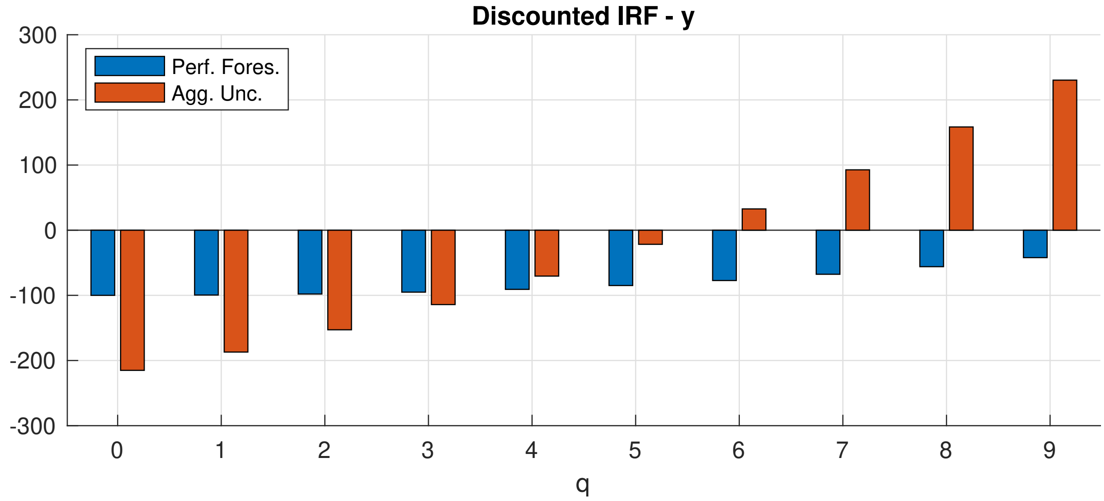
presence of idiosyncratic risk strongly amplifies the interaction between aggregate uncertainty and the ZLB.

6 Unconventional Monetary Policy and HANK

In this section we perform a policy exercise that the model and our solution strategy permits: we study the effect of forward guidance in our HANK model and the differential effects between the stochastic and deterministic environment. This analysis, as the one highlighted in figure 9 requires multiple simulations of our model, which is rendered feasible by our solution methodology.

We consider the following forward guidance policy. The central bank credibly announces that it will set the nominal interest rate to 0 for q additional quarters, relative to what would be implied by the Taylor rule. The extra stimulus q is unconditional on the specific contingency realization, implying that this policy increases the expected duration of the liquidity trap exactly by q quarters. This analysis is similar to what McKay, Nakamura and Steinsson (2016) label as extended policy (where they choose q to minimize output loss on impact in a RANK economy) and to some extent goes in the direction of the state-contingency mentioned by Woodford (2012). Under the deterministic shock, such policy also corresponds to the "fixed

Figure 12: Discounted IRF and Forward Guidance - Output



Notes: The figure reports the implied discounted impulse response functions for output under the HANK model with the ZLB, in the forward guidance experiment. The order of the columns corresponds to the quarters of extra stimulus under the forward guidance policy. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

length forward guidance" policy in Eggertsson et al. (2021). However, the equivalence does not hold with the stochastic shock.

Figure 12 shows the effects, on the discounted impulse response of output, of forward guidance under the deterministic (blue bars) and stochastic (red bars) shocks as a function of the q quarters of extra stimulus.[33] All bars are relative to the no forward guidance policy under the deterministic case (in fact, the first column in the figure is exactly the same as the first column in figure 11). The plot reveals that forward guidance is more effective under the stochastic case: with the calibrated shock, it takes 6 quarters of extra stimulus to actually flip the output loss to an output gain. The same does not happen under the deterministic shock, despite the fact that the output loss is smaller under the deterministic shock than under the stochastic shock to begin with. A similar result holds for inflation (we report the figure in the appendix).

7 Conclusions

We develop a novel methodology to solve heterogeneous agents models with aggregate uncertainty and a Zero Lower Bound on nominal interest rates. By considering a Two-states Markov shock structure as in Eggertsson et al. (2021), we are able exploit and expand on the tech-

[33] See the Appendix for the analogue figure for inflation.

niques proposed by Auclert et al. (2021). The efficiency and flexibility of our methodology let us consider several counterfactual scenarios.

Our main application involves studying the interaction of aggregate and idiosyncratic uncertainties. We show that at the Zero Lower Bound, aggregate uncertainty amplifies a demand shock, and this amplification is much stronger if we consider a heterogeneous agent economy. If however the monetary authority is unconstrained, no amplification takes place. We illustrate the mechanisms behind these results with a simple model that accommodates an analytical solution.

We exploit our solution methodology to study the impact of forward guidance. The model simulations indicate that the marginal effects of a promise to keep interest rates at 0 for an extra quarter are larger when there is aggregate uncertainty.

We hope that our methodology allows future researchers to better understand the interactions between uncertainty in the micro and macro level, the role of other types of policy such as government transfers, as well as the impact of other non-linearities at the macro level such as occasionally binding constraints in the financial sector (as in, e.g., Gertler and Kiyotaki (2015)). Our solution strategy can also be employed to evaluate monetary policy normalization in the current economic environment, characterized by large uncertainty both at the aggregate and individual levels.

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Appendix

A Notation List

- X_t and X_t^τ are vectors of 5 entries (GDP, inflation, R, t, b)
- D_t and D_t^τ are vectors of nz x na entries, representing the distribution AT THE BEGINNING of the t period
- V_t and V_t^τ are vectors of nz x na entries, representing the value function AT THE BEGINNING of the t period
- $\mathbb{X}^{TS}$ is a stacked vector made of $\tau^{\max}$ vectors of 5 entries representing inputs along the TS branch. In loose sense $\mathbb{X}^{TS} = \{X_t\}_{t=0}^{\tau^{\max}-1}$
- $\mathbb{D}^{TS}$ is a stacked vector made of $\tau^{\max}$ vectors of $n_a \times n_z$ entries representing the distribution at the beginning of each period along the TS branch. In loose sense $\mathbb{D}^{TS} = \{X_t\}_{t=0}^{\tau^{\max}-1}$
- $\mathbb{V}^{TS}$ is a stacked vector made of $\tau^{\max}$ vectors of na x nz entries representing the value functions the TS branch. In loose sense $\mathbb{V}^{TS} = \{V_t\}_{t=0}^{\tau^{\max}-1}$
- $\mathbb{X}^\tau$ is a stacked vector made of T vectors of 5 entries representing inputs along one of the PF branches. In loose sense $\mathbb{X}^\tau = \{X_t^\tau\}_{t=\tau}^{T+\tau-1}$
- $\mathbb{D}^\tau$ is a stacked vector made of T vectors of $n_a \times n_z$ entries representing the distribution at the beginning of each period of the perfect foresight branch. In loose sense $\mathbb{D}^\tau = \{D_t^\tau\}_{t=\tau}^{T+\tau-1}$
- $\mathbb{V}^\tau$ is a stacked vector made of $\tau^{\max}$ vectors of na x nz entries representing the value functions in one of the PF branches. In loose sense $\mathbb{V}^\tau = \{V_t^\tau\}_{t=\tau}^{T+\tau-1}$
- $\mathbb{X}^{PF} = \{\mathbb{X}^\tau\}_{\tau=1}^{\tau^{\max}}$
- $\mathbb{D}^{PF} = \{\mathbb{D}^\tau\}_{\tau=1}^{\tau^{\max}}$
- $\mathbb{V}^{PF} = \{\mathbb{V}^\tau\}_{\tau=1}^{\tau^{\max}}$
- $\mathbb{X}_1^{PF} = \{X_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$ is the collection $5x1$ vectors of inputs in the first period of *each* PF path.
- $\mathbb{V}_1^{PF} = \{V_\tau^\tau\}_{\tau=1}^{\tau^{\max}}$

B Technical Details

B.1 Aggregate State Variables

One of the arguments in equation (25) is the realization of aggregate variables $X_{\tau-1}$, during the period right before the contingency is revealed. Those are effectively initial conditions for the τ-th PF branch under consideration. Given the dynamic programming structure for the households' problem, those initial conditions do not enter the Heterogeneous Agent block. However, they might enter some aggregate equilibrium conditions. One example is the stock of public debt $b_{\tau-1}$. This is an initial condition that should be taken into account under a more general fiscal policy rule. We account for the effects of aggregate state variables in equation (25) by deriving the (analytical) Jacobian of aggregate equilibrium conditions with respect to these variables.

B.2 Lags

In our model economy, there were no significant variables that entered with a lag larger than 1. This might not be true for more complex models. For instance, if one was to study the new Average Inflation targeting framework of the Federal Reserve, the model should keep track of many past levels of inflation. And in particular, once entering a contingency τ, it would not be enough to carry over the information in $X_{\tau-1}$. The solution is to define an aggregate variable which at time t takes the value of the lag variable of interest. To give a practical example, if the model requires to keep track of inflation 2 periods in the past, define $\Pi_{Lag2,t} = \Pi_{t-2}$ and use this as another structural equation.

B.3 Leads

The story is slightly more complicated for leads. Some models might require to form expectations of future variables with lead larger than 1. Suppose that you are interested in considering in the equilibrium conditions the expectations for a variable x in l quarters in the future. The solution is to define l auxiliary variables as follows. $AUX1_t = \mathbb{E}_t AUX2_{t+1}$, $AUX2_t = \mathbb{E}_t AUX3_{t+1}$,..., $AUXl_t = \mathbb{E}_t x_{t+1}$. The variable $AUX1_t$ will be the one that in the equilibrium conditions substiutes $\mathbb{E}_t x_{t+l}$.

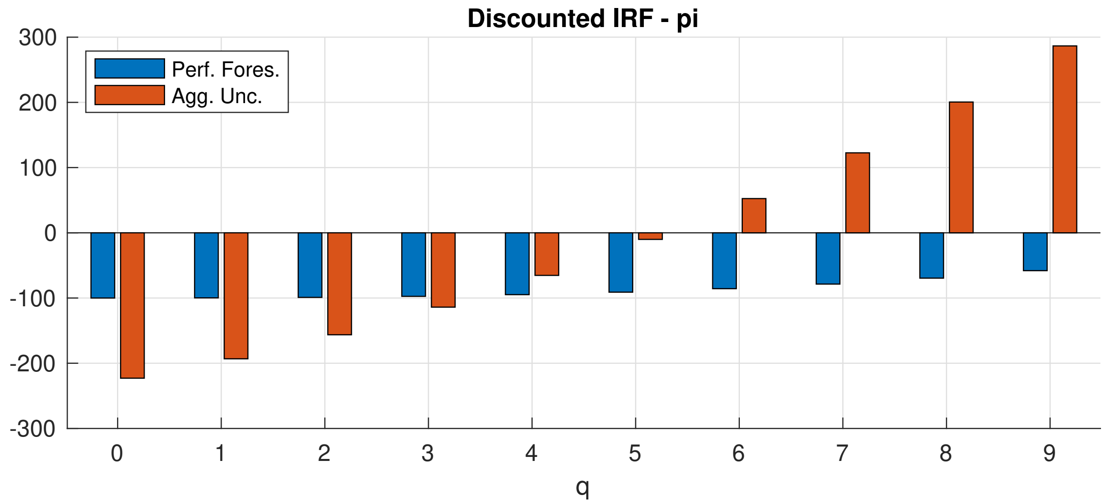
For a concrete example, consider inflation 2 periods ahead. This example only considers the TS branch, as for any PF branch the problem is trivial. Define $AUX1_t = \mathbb{E}_t AUX2_{t+1} = \mu AUX2_{t+1} + (1-\mu)AUX2_{t+1}^{t+1}$ and $AUX2_t = \mathbb{E}_t \Pi_{t+1} = \mu \Pi_{t+1} + (1-\mu)\Pi_{t+1}^{t+1}$. We want to show that $AUX1_t$ is the correct expectation of inflation in 2 periods. Consider for simplicity time $t = 0$, we want that $AUX1_0 = \mathbb{E}_0 \Pi_2$.

$$\begin{aligned}
AUX1_0 &= (1-\mu)AUX2_1^1 + \mu AUX2_1 \\
&= (1-\mu)\Pi_2^1 + \mu(1-\mu)\Pi_2^2 + \mu\mu\Pi_2
\end{aligned}$$

C Other Figures

Figure C.1 reports the discounted impulse response functions of inflation as a function of the quarters of forward guidance, as described in Section 6. The bars are normalized with respect to the deterministic shock with no forward guidance (i.e. $q = 0$).

Figure C.1: Discounted IRF and Forward Guidance - Inflation

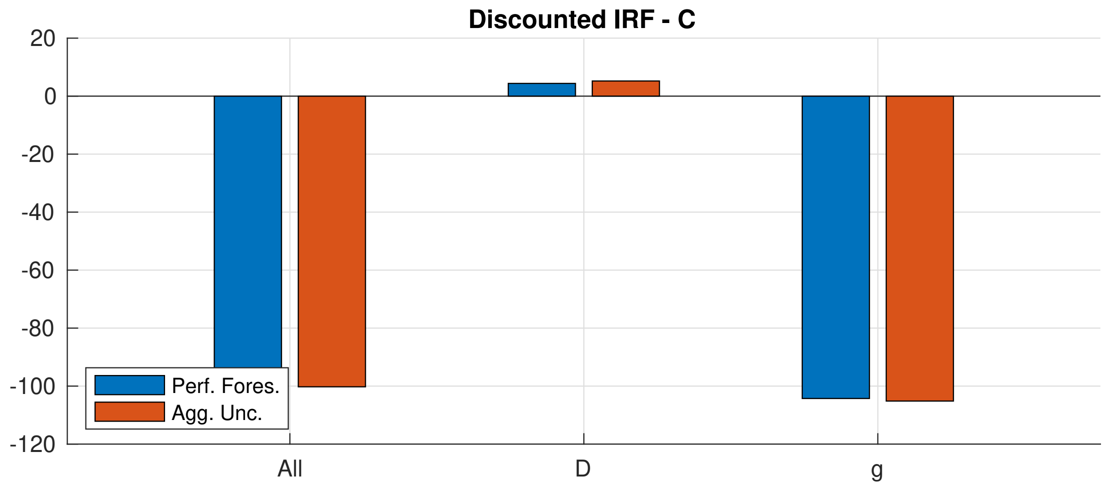


Notes: The figure reports the implied discounted impulse response functions for inflation under the HANK model with the ZLB, in the forward guidance experiment. The order of the columns corresponds to the quarters of extra stimulus under the forward guidance policy. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

Figure C.2 reports the decompositions for the discounted impulse response functions of total consumption. We feed either the equilibrium wealth distribution (middle column) or all the equilibrium prices (right column) into the individual policy functions of the households. The first case, that is equivalent to applying the steady state consumption policy function across the equilibrium wealth distributions, isolates the effects of the latter ones. The second case does the reverse, repeatedly applying the equilibrium consumption policy functions to the steady state wealth distribution and isolating the effect of the equilibrium prices.

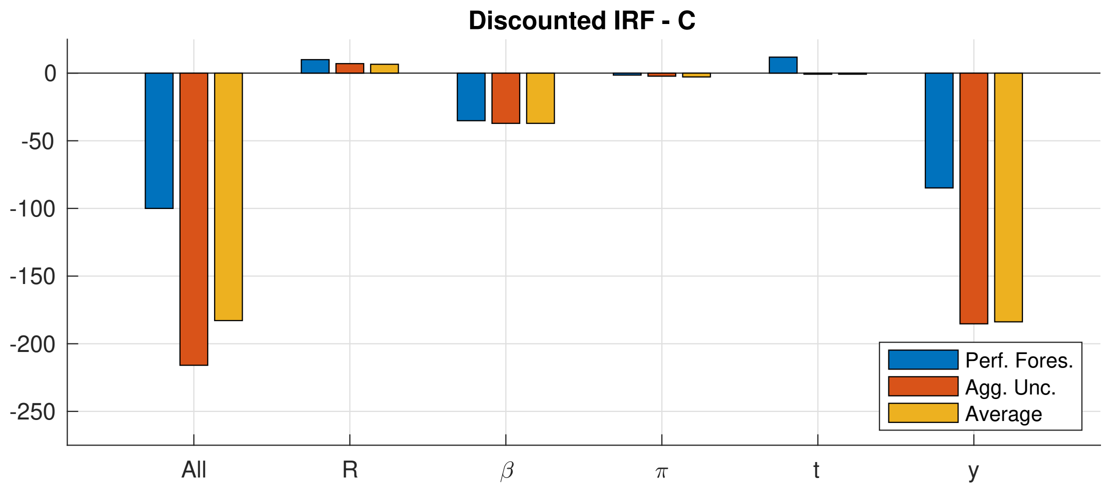
In a similar fashion, Figure C.3 decomposes the impulse response function of total consumption by applying one aggregate price or shock at a time. The yellow bar correspond to a counterfactual experiment where agents live in a deterministic world and are given the average aggregate dynamics obtained in the aggregate uncertainty case as inputs.

Figure C.2: Discounted IRF - Decomposition - D and g



Notes: The figure reports the implied discounted impulse response functions for consumption under the HANK model with the ZLB. The columns correspond to the full effects, the effects of the distribution, and the effects of the individual policies. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

Figure C.3: Discounted IRF - Decomposition - Prices



Notes: The figure reports the implied discounted impulse response functions for consumption under the HANK model with the ZLB. The columns correspond to the full effects, the effects of nominal rate, discount factor, inflation, taxes, incomes. Within each column, the blue (red) bar corresponds to the deterministic (stochastic) case. The yellow bar corresponds to a deterministic counterfactual where agents are given the average of the prices in the stochastic case. All bars are relative to the one in the HANK model with the ZLB, under the deterministic shock and with no extra stimulus (left most blue bar).

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