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Demographics and the natural real interest rate: historical and projected paths for the euro area

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Abstract

This paper employs an aggregate representation of an overlapping generation (OLG) model quantifying a decrease of the natural real interest rate in the range of -1.7 and -0.4 percentage points in the euro area between 1990 and 2030 due to demographics alone. Two channels contribute to this downward impact: the increasing scarcity of effective labor input and the increasing willingness to save by individuals due to longer life expectancy. The decrease of the aggregate saving rate as individuals retire has an upward impact which is never strong enough. Mitigating factors are: higher substitutability between labor and capital, higher intertemporal elasticity of substitution in consumption, reforms aiming at increasing the relative productivity of older cohorts, the participation rate and the retirement age. The simulated path of the natural real interest rate is consistent with recent econometric estimates: an upward trend in the 70s and 80s and a prolonged decline afterward.

JEL codes: E17, E21, E43, E52, J11.

Keywords: demographic transition, aging, natural interest rate, secular stagnation, euro area

We have this unusual degree of knowledge concerning the future because of the long but definite time-lag in the effects of vital statistics. Nevertheless the idea of the future being different from the present is so repugnant to our conventional modes of thought and behaviour that we, most of us, offer a great resistance to acting on it in practice. (Keynes (1937), Some Economic Consequences of a Declining Population)

A temporary period of policy rates being close to zero or even negative in real terms is not unprecedented by any means. Over the past decades, however, we have seen long-term yields trending down in real terms as well, independent of the cyclical stance of monetary policy. (Draghi (2016), Addressing the Causes of Low Interest Rates)

Non-technical summary

Real interest rates have been on a downward trend since the late 80s. This observation leads to ask whether the natural interest rate – the unobservable real interest rate that, following Wicksell (1898), brings output in line with its potential and tends to neither accelerate nor decelerate prices – has decreased as well and whether it will remain low in the years ahead. Among the factors that could explain the downward trend, researchers have been recently looking for potential "slow-moving secular forces" (Eggertsson, Mehrotra, and Robbins, 2017). Demographic change is one of such forces. How it can impact the natural interest rate is the exclusive focus of this paper.

Advanced economies are undergoing an aging process, particularly pronounced in Europe, characterized by decreasing fertility and mortality rates. Before the 80s, the ratio of the elderly (aged 65 and above) to working-age (aged 15-64) was less than 2 to 10, in 2050 the proportion will be more than 5 to 10 according to UN (2017)'s projections. Demographic projections several years into the future tend to be reliable due to unequivocal time-lags characterizing the demographic evolution.

Within the neoclassical framework, overlapping generations (OLG) models are considered the best tool to capture the impact of demographic change as they allow to use the empirical age distribution in a context of a life-cycle behavior. Since the goal is to study long-term 'natural' trends, in the paper I consider an OLG model that abstracts from nominal frictions, market imperfections and shocks other than the ones implied by demographics. In this model, as it is standard, the natural interest rate is the real rate of return on capital (net of depreciation) that allows the saving supply (by households) to meet the capital demand (by firms). Throughout the paper it is shortly called real interest rate. Then, revising a methodology first employed by Jones (2018) that allows to approximate the solution of the optimization problem of an OLG model via a representative agent's setup, I find an aggregate representation that sheds light on the channels through which demographic change can affect the real interest rate and I provide quantitative estimates. The key-driving variable turns out to be the growth rate of the effective labor-population ratio, that is the ratio of the number of people in the working-age evaluated according to the age-dependent productivity over the number of people in the entire population. The growth rate of this ratio decreases by about 1 percentage point between 1990 and 2030 according to UN (2017)'s data, while it was positive and increasing in the 70s and 80s for the euro area.

There are three channels through which aging affects the real interest rate in a closed-economy. (1) *Downward impact from lower labor input*. A decrease of the growth rate of the effective-labor population ratio is akin to a slowdown in total factor productivity for output per capita growth, which leads firms to demand less capital, reducing the marginal product of capital and so the real interest rate, everything else being equal. This effect is stronger the more the age-distribution shifts towards older cohorts that are parametrized to be less productive and the lower the degree of substitutability in production between capital and labor. (2) *Downward impact from higher life expectancy*. Assuming perfect foresight, the representative household realizes that the growth rate of the number of effective workers in support of the number of total consumers (the population size) is shrinking over time. Therefore, with the goal of smoothing consumption per capita into the future and depending on the pension scheme in place, the representative household anticipates this change with a willingness of consuming less and saving more, i.e. becoming more patient thus decreasing the real interest rate, everything else being equal. This effect is stronger the higher the growth rate of the average survival probability in the economy and the lower the intertemporal elasticity of substitution in consumption. (3) *Upward impact from a rising proportion of dissavers*. As the growth of the effective labor force decreases, it is optimal to reduce the pace of capital accumulation in support of labor. Hence, the aggregate investment rate (and so the saving rate) decreases with aging leading *ceteris paribus* to an increase of the real interest rate. In this way, the model captures that as the growth rate of the number of those who save (workers) compared to the universe of consumers (population) decreases, the economy generates smaller saving rates and so higher real interest rate, everything else being equal.

I calibrate the model for the euro area using demographic data and projections by UN (2017) as exogenous variations to study a perfect-foresight transition where the demographic change is perfectly anticipated by the agents in the economy. The quantitative exercise shows that the real interest rate path guided by demographic change exhibits a rise throughout the 70s and 80s, and then a prolonged fall at least until 2030. The shape is consistent with low-frequency econometric estimates (cf. WGEM (2018)). In the baseline analysis the real interest rate declines between about 1 and 1.25 percentage points going from 1990 to 2030 (roughly the peak to trough in the simulation). Under the various sensitivity exercises the range of estimates suggests a decrease standing between -1.7 and -0.4 percentage points. These exercises show that the dampening effect of aging could be mitigated not only by higher substitutability between labor and capital and higher elasticity of intertemporal substitution in consumption, but also by reforms aiming particularly at increasing the relative productivity of older cohorts and the participation rate. An increase of the retirement age with no other supporting reform leads to a higher path of the real interest rate, but only in a limited and quantitatively insignificant extent. Isolating the quantitative impact of the different channels shows that as aging unfolds channels (1) and (2) contribute almost equally to the dampening effect on the real interest rate while channel (3) is never strong enough.

1 Introduction

Advanced economies are undergoing a demographic transition, the ageing process by which "populations move from initially high fertility and mortality with young age distributions to low fertilty and mortality with old age distributions", cf. Lee (2016). Figures 1 and 2 show this process for Europe: as the number of people entering the world is shrinking and mortality rates are decreasing (i.e. the survival probability is increasing), the relative number of the elderly is dramatically increasing. The process is very similar across different European geographic areas, no matter if one considers the euro area composed by 19 countries, the big 5, the core 12, or the wide European Union composed by 28 countries. While before the 80s the ratio of the elderly (aged 65 and over) to working age (aged 15-64) has been less than 2 to 10, the United Nations (UN, 2017) project this proportion to rise above 5 to 10 by year 2050 in Europe. These figures have been known for decades as demographic projections tend to have low uncertainty. It is therefore appealing to use demographic data as exogenous variation to explain macroeconomic dynamics.

Questioning the influence of demographic change on the real interest rate is certainly not new in economic research. What is new in recent years is that the topic is on the agenda of central bankers. The fact that real interest rates have been on a downward trend since the late 80s across many countries leads to ask whether the natural interest rate has decreased as well and whether it will remain low in the years ahead, potentially hampering the effectiveness of monetary policy. The definition of natural or neutral interest rate dates back to Wicksell (1898)[1] and in modern macroeconomics can be thought as the rate of interest that brings output in line with its potential or natural level in the absence of transitory shocks or nominal adjustment frictions, thus tending to stabilize the dynamics of prices. It will be identified as the real rate of return on capital (net of depreciation) that allows the saving supply (by households) to meet the capital demand (by firms) in absence of any allocational friction or arbitrage. The *real* macro model employed in the paper will abstract from frictions and shocks that could capture business cycle variations. The natural rate of return on capital will be shortly called real interest rate.

Researchers have been recently looking for potential "slow-moving secular forces" as explanatory factors behind the downward trend in real interest rates (cf. Eggertsson, Mehrotra, and Robbins (2017)). Demographic change is one of such forces and how it can affect the natural real interest rate, an *unobservable* variable, is the exclusive focus of this paper.

In a standard Solow (1956)'s model with homogeneous population and a constant saving rate, as population growth decreases capital per worker rises dampening the marginal product of capital, so that in equilibrium the more abundant factor, capital, receives a lower remuneration than the other factor, labor: the real interest rate falls while the real wage rises. Models embedded in the neoclassical framework, like general equilibrium overlapping generations (OLG) models, can rarely escape the prediction that ageing leads to a lower real interest rate.

Since the seminal contribution by Auerbach and Kotlikoff (1987), OLG models are considered the most reli-

[1] *"There is a certain rate of interest on loans which is neutral in respect to commodity prices, and tends neither to raise nor to lower them. This is necessarily the same as the rate of interest which would be determined by supply and demand if no use were made of money and all lending were effected in the form of real capital goods. It comes to much the same thing to describe it as the current value of the natural rate of interest on capital". (Wicksell, 1898)*

able tool to evaluate the macroeconomic effect of demographic change as they allow to use the full empirical age distribution, in a context of a flexible life-cycle behavior. Recent contributions employing fully-fledged OLG models all predict a downward trend of the real interest rate due to population aging, no matter if the model encompasses the whole world economy with different countries/areas (cf. e.g. Domeij and Floden (2006), Krueger and Ludwig (2007)) or a single country/area modeled as closed-economy (cf. e.g. Gagnon, Johannsen, and Lopez-Salido (2016) for US, Bielecki, Brzoza-Brzezina, and Kolasa (2018) for Europe, Sudo and Takizuka (2018) for Japan). The difficulty with these models is that the computational challenge involved makes it hard to isolate the mechanisms through which aging affects the real interest rate.

With the main goal of enlightening these mechanisms but still providing a reliable quantitative estimate, I revise a methodology first employed by Jones (2018) that is shown to well approximate the solution paths of an OLG model that crucially assumes the existence of a "perfect annuity market" by solving only for the aggregates.[2] Using this methodology, the OLG model has an aggregate representation where the demographic change is captured by perfectly anticipated time-varying parameters that keep track of the evolving age-structure in each period of the transition within a representative agent framework. Thanks to this representation I show that the growth rate of the *effective labor-population ratio* is the key variable to understand the impact of demographic change throughout the transition. It depends not only on the number of workers but also on their age-dependent productivity, as compared to the number of people in the whole economy. According to UN (2017) data and projections, it is positive and increasing in the 80s, continuously decreasing afterward (a decrease of about 1 percentage point between 1990 and 2030).

There are three channels through which aging affects the real interest rate in a closed-economy (cf. Krueger and Ludwig (2007), and Carvalho, Ferrero, and Nechio (2016)). (1) *Downward impact from lower labor input*. A decrease of the growth rate of the effective-labor population ratio is akin to a slowdown in total factor productivity for output per capita growth, which leads firms to demand less capital, reducing the marginal product of capital and so the real interest rate, everything else being equal. This effect is stronger the more the age-distribution shifts towards older cohorts that are parametrized to be less productive and the lower the degree of substitutability in production between capital and labor. (2) *Downward impact from higher life expectancy*. Assuming perfect foresight, the representative household realizes that the growth rate of the number of effective workers in support of the number of total consumers (the population size) is shrinking over time. Therefore, with the goal of smoothing consumption per capita into the future and depending on the pension scheme in place, the representative household anticipates this change with a willingness of consuming less and saving more, i.e. becoming more patient thus decreasing the real interest rate, everything else being equal. This effect is stronger the higher the growth rate of the average survival probability in the economy and the lower the intertemporal elasticity of substitution in consumption. (3) *Upward impact from a rising proportion of dissavers*. As the growth of the effective labor force decreases, it is optimal to reduce the pace of capital accumulation in support of labor. Hence, the aggregate investment rate (and so the saving rate) decreases with aging leading *ceteris paribus* to an increase of the real interest rate. In this way, the

[2]The assumption is that agents within each age group agree to share equally the assets of the dying members of their age group (unintentional bequest) among the surviving members. Therefore, agents are perfectly insured against mortality risk.

model captures that as the growth rate of the effective number of those who save (workers) compared to the universe of consumers (population) decreases, the economy generates smaller saving rates and so higher real interest rate, everything else being equal.

I calibrate the model for the euro area using demographic data and projections by UN (2017) as exogenous variations to study a perfect-foresight transition where the demographic change is perfectly anticipated by the agents in the economy. The quantitative exercise shows that the real interest rate path guided by demographic change exhibits a rise throughout the 70s and 80s, and then a prolonged fall at least until 2030. The shape is consistent with low-frequency econometric estimates of the natural real interest rate based on a large time span that also highlight the importance of demographic factors (cf. WGEM (2018)). In the baseline analysis the real interest rate declines between about 1 and 1.25 percentage points going from 1990 to 2030 (roughly the peak to trough in the simulation). Under the various sensitivity exercises the range of estimates suggests a decrease standing between -1.7 and -0.4 percentage points. These exercises include variations in the capital-labor elasticity of substitution, the intertemporal elasticity of substitution in consumption, extensions to endogenous labor supply, different retirement ages, pension schemes and age-dependent productivity levels. They show that the dampening effect of aging could be mitigated not only by higher substitutability between labor and capital and higher intertemporal elasticity of substitution in consumption, but also by reforms aiming particularly at increasing the relative productivity of older cohorts and the participation rate. An increase of the retirement age with no other supporting reform leads to a higher path of the real interest rate, but only in a limited and quantitatively insignificant extent. Finally, the model predicts that, since aging leads to a steady decrease of the investment rate, there is a reallocation of labor away from the sector whose goods are used for capital investment towards the more services-oriented sector.

Given that as the aging process unfolds the real interest rate decreases, the model predicts that channels (1) and (2) above are stronger than channel (3). A decomposition of these channels shows that the channels hold with the expected sign in the projected period, but channel (3) is never strong enough to counteract the other two which add up with almost an equal contribution to the dampening effect of aging on the real interest rate. These results cast doubts on predictions based on the fact that "old dis-save, while workers save. The more old people there are, the less saving there will be" (Goodhart and Pradhan, 2017) to suggest an increase of the real interest rate due to aging. Instead, they tend to support corroborated results from OLG models where, as noted also by Lisack, Sajedi, and Thwaites (2017), the lack of reversal of the real interest rate due to aging can be attributed to the prevalence of effects from the stock of wealth relative to the flow of dissavings generated by retiring baby boomers.

This paper is mostly related to Krueger and Ludwig (2007), Ikeda and Saito (2014), Kara and von Thadden (2016), Carvalho, Ferrero, and Nechio (2016), Jones (2018). As mentioned above, I rely on Jones (2018)'s methodology to find a suitable aggregate representation. The focus of his analysis is more on the recent deviation of output growth from its long-run trend in the US including business cycle fluctuations and evaluating the role of the zero lower bound for monetary policy. Carvalho, Ferrero, and Nechio (2016), who start the analysis in 1990, also highlight three channels through which the demographic transition can affect the real interest rate, finding that increasing expected life expectancy (channel (2)) is almost uniquely responsible for declining real interest rate (in a representative OECD economy). Their tractable setting *à la* Gertler (1999) might overstimate the life-expectancy channel as congectured by Gagnon, Johannsen, and Lopez-Salido (2016) who, on the contrary, find that the contribution of longer life expectancy has a much smaller impact (for the US). Using a similar tractable setting calibrated for the euro area, Kara and von Thadden (2016) find an impact on the real interest rate due to demographics which is comparable in magnitude to the one I found in the baseline, despite their simulation starts in the 2000s. Ikeda and Saito (2014) highlights the importance of the working-age population ratio as the main driver in a aggregate model which, however, is not linked to an OLG setting (and is calibrated for Japan). Krueger and Ludwig (2007), one of the most comprehensive contributions on the subject, highlight some channels on a balanced-growth path of the model with constant effective capital stock while the results from their full model are provided only from the 2000s.

The rest of the paper proceeds as follows. Section 2 presents the OLG model and how it can be approximated with an aggregate representation. Section 3 shows the analytics to solve the model and explains the theoretical channels through which the demographic transition can impact the real interest rate. Section 4 presents the quantitative simulations based on the model calibrated for the euro area. Section 5 concludes.

2 Model

2.1 Households

Consider a standard overlapping generations (OLG) model for a closed economy. Each household consists of a single individual. Households within each cohort j are identical and their exogenous mass $N_{t,j}$ for time-period t evolves recursively according to:

$$N_{t,j} = N_{t-1,j-1} s_{t,j} \tag{2.1}$$

where $s_{t,j}$ is the conditional survival probability.[3] It is assumed that households enter the world as workers at the age of 15 ($j = 0$) and remain alive up to age 100 ($j = J = 85$), after that they die with certainty. So the economy is populated by $J + 1$ cohorts of overlapping generations.

A representative j–aged household maximizes the utility function choosing consumption and the amount of assets to hold the following period for each life period, $c_{t+j,j}, a_{t+j+1,j+1}$. Assuming a CRRA utility function, the maximization problem is:

$$\max_{c_{t+j,j}, a_{t+j+1,j+1}} \sum_{j=0}^{J} \beta^j \pi_{t+j,j} \frac{(c_{t+j,j})^{1-\sigma}}{1-\sigma}$$

[3] Given that an individual is aged $j-1$ at time $t-1$, $s_{t,j}$ is the probability to be alive at age j at time t. Following Domeij and Floden (2006), data are taken for $N_{t,j}$ for all available t, j to get the implied survival probabilities $s_{t,j}$ which therefore can exceed 1 due to migration flows. The underlying assumption is that immigrants arrive without assets and are adopted by domestic households.

subject to

$$\begin{aligned}
a_{t+j+1,j+1} &= \frac{a_{t+j,j}(1+r_{t+j})}{s_{t+j,j}} - c_{t+j,j} + y_{t+j,j} \\
y_{t+j,j} &= (1-\tau_{t+j})w_{t+j}h_{t+j,j}I(j \leq jr) + d_{t+j,j}I(j > jr) \\
a_{t+J+1,J+1} &= 0 \\
a_{t,0} &= 0
\end{aligned}$$

where $\pi_{t+j,j} = \prod_{k=0}^{j} s_{t+k,k}$ is the unconditional survival probability with $s_{t,0} = 1$; β is the discount factor; σ is the coefficient of risk-aversion; r_{t+j} is the real interest rate; w_{t+j} is the real wage; τ_{t+j} is the tax rate on labor income; $I(\cdot)$ is an indicator function; jr denotes the last working age (so that $jr+1$ is the first period of retirement), exogenously imposed; $h_{t+j,j} = h_j$ for all t is an exogenously given amount of hours to work depending on age, constant over time (cf. Figure 3); $d_{t+j,j}$ is the pension transfer from the government. Each household is born with zero wealth and is not allowed to die with either positive or negative wealth. It is assumed that there exist a "perfect annuity market".[4]

Jones (2018) shows that the solution of the maximization problem above can be well approximated by solving the following social planner's problem (which allows to solve for the aggregates in the model without knowing the distribution of wealth across individuals, thus simplifying the computational challenge):

$$\begin{aligned}
&\max_{C_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \varphi_t \frac{C_t^{1-\sigma}}{1-\sigma} \\
\text{s.t.} \qquad &C_t + K_t = (1-\tau_t)w_t L_t + (1+r_t)K_{t-1} + T_t
\end{aligned}$$

where all variables are real aggregates. Specifically, C_t is consumption; K_t is capital; T_t denotes any transfer received by the representative agent; L_t is the exogenous labor supply that by clearing in the labor market reads:

$$L_t = \sum_{j=0}^{J} h_j N_{t,j} \tag{2.2}$$

The time-varying parameter φ_t allows to represent the (finite–life) individuals' problem of the OLG setting as

[4]The assumption of "perfect annuity market" means that the agents within each age group j agree to share the assets of the dying members of their age group among the surviving members. Using the notation just introduced, consider those that at time t are aged j. The total amount of assets of the dying members is: $a_{t,j}(1-s_{t,j})N_{t-1,j-1}$, while the number of surviving members is: $N_{t,j} = N_{t-1,j-1}s_{t,j}$. Hence, in the budget constraint the asset holding in period $t+1$ will depend on what as been accumulated plus this sort of 'equal gift' from the dying members given the real interest rate (r_t) at which these assets can be invested (minus consumption plus income):

$$\begin{aligned}
a_{t+1,j+1} &= a_{t,j}(1+r_t) + \frac{a_{t,j}(1+r_t)(1-s_{t,j})N_{t-1,j-1}}{N_{t-1,j-1}s_{t,j}} - c_{t,j} + y_{t,j} \\
&= \frac{a_{t,j}(1+r_t)}{s_{t,j}} - c_{t,j} + y_{t,j}
\end{aligned}$$

which is the budget constraint written in the main text.

a problem of an infinitely–lived representative agent (i.e social planner). It captures the size of the population in each period, scaled by the coefficient of risk-aversion:

$$\varphi_t \quad = \quad \left[\sum_{j=0}^{J} N_{t,j}(\lambda_j)^{\frac{1}{\sigma}}\right]^{\sigma} \overset{\lambda_j=1}{=} N_t^{\sigma} \tag{2.3}$$

where λ_j are the welfare weights attached to each individual of age j which, following Jones (2018), are set to be equal across all individuals. For simplicity, $\lambda^j = 1$ for all j. Hence, $\varphi_t = N_t^{\sigma}$ where $N_t = \sum_{j=0}^{J} N_{t,j}$ is the number of people in the economy in each period (between age 15 and 100). Jones (2018) adds an ad-hoc multiplicative shock ζ_t to the discount factor β that captures the change in average longevity in the economy even though this shock does not appear in the derivation that maps the decentralized equilibrium into the social planner's allocation. In the remainder of the paper the discount factor will be $\beta\zeta_t$ and the role of ζ_t will be evaluated.

Connecting this work to Giagheddu and Papetti (2018) and Papetti (2018), the setting above is slightly modified introducing two sectors that are thought as tradable (T) versus non-tradable (N) even though the environment is for a closed-economy. What is important, beyond labels, is that only T-goods can be used for the purpose of investment in physical capital. The T-good will be the numeraire. Thus, the representative household will choose consumption in the two sectors C_t^T, C_t^N for given relative price $Z_t \equiv P_t^N/P_t^T$ and additionally how to allocate in the two sectors the exogenous amount of hours to work L_t: L_t^T, L_t^T, given the exogenous sectoral real wages, w_t^N, w_t^T. In the general formulation, imperfect substitutability of working hours between the two sectors is assumed via a CES aggregator: the case of $\epsilon \to \infty$ corresponds to the case of perfect substitutability (or perfect mobility), while $\epsilon \to 0$ corresponds to immobility. Hence, assuming a Cobb-Douglas consumption aggregator with $0 < \gamma < 1$, the representative household's problem is the following:

$$\max_{C_t^T, C_t^N, L_t^T, L_t^T, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \zeta_t \varphi_t \frac{\left[\left(C_t^T\right)^{\gamma}\left(C_t^N\right)^{1-\gamma}\right]^{1-\sigma}}{1-\sigma} \tag{2.4}$$

$$\text{s.t.} \qquad C_t^T + Z_t C_t^N + K_t = (1-\tau_t)(w_t^T L_t^T + w_t^N L_t^N) + (1+r_t)K_{t-1} + T_t \tag{2.5}$$

$$L_t = \left[\chi^{-\frac{1}{\epsilon}}(L_t^T)^{\frac{\epsilon+1}{\epsilon}} + (1-\chi)^{-\frac{1}{\epsilon}}(L_t^N)^{\frac{\epsilon+1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon+1}} \tag{2.6}$$

Notice that to have this representation one needs to prove that the problem of choosing sectoral working hours for each individual can be represented as a problem of choosing aggregate sectoral hours for the social planner. This is done in Appendix A.

2.2 Government

Given a certain level of generosity of the PAYGO pension system, i.e. the replacement rate $\bar{d}$ defined as the pension benefit d_t received by each household per unit of the average labor income $w_t(1-\tau_t)\bar{h}$, the government sets a tax rate τ_t such that its budget is balanced in each period:

$$d_t = \bar{d} w_t (1-\tau_t)\bar{h} \tag{2.7}$$

$$\tau_t w_t L_t = d_t \sum_{j=jr+1}^{J} N_{t,j} = T_t \tag{2.8}$$

where w_t is the economy-wide hourly real wage such that $w_t L_t = w_t^T L_t^T + w_t^N L_t^N$ and $\bar{h} = (\sum_{j=0}^{jr} h_j)/(jr+1)$ is the average efficiency-hours worked in the working life-periods.[5] The unique transfer to the representative household is the aggregate pension benefit $T_t = d_t \sum_{j=jr+1}^{J} N_{t,j}$. However, notice that in the current setting with exogenous labor supply the government is uninfluential on agents' choices.

2.3 Firms

Production happens under perfect competition with constant-returns to scale technology which is a Cobb-Douglas production function in the baseline setting:[6]

$$Y_t^T = (K_t^T)^{\psi}(A_t^T L_t^T)^{1-\psi} \tag{2.9}$$

$$Y_t^N = (K_t^N)^{\psi}(A_t^N L_t^N)^{1-\psi} \tag{2.10}$$

where $0 < \psi < 1$ is the capital elasticity of output assumed to be equal between sectors; A_t^T, A_t^N are the exogenous labor-augmenting technological parameters in the two sectors. The factor markets are also perfectly competitive. Therefore, one can consider a representative firm in each sector T, N hiring (efficiency units of) labor L_t^T, L_t^N at a given hourly real wage w_t^T, w_t^N and renting capital K_t^T, K_t^N at the rental rate r_t^k subject to the yearly depreciation rate δ. Each representative firm maximizes per-period profits:

$$\max_{K_t^T, L_t^T} \left\{ Y_t^T - w_t^T L_t^T - r_t^k K_t^T \right\} \tag{2.11}$$

$$\max_{K_t^N, L_t^N} \left\{ Z_t Y_t^N - w_t^N L_t^N - r_t^k K_t^N \right\} \tag{2.12}$$

2.4 Financial intermediary

There exists a financial intermediary that ensures no arbitrage in the rental market for capital by making zero profits.[7] Consider the underlying timing and dynamics. Throughout period $t-1$ the representative household saves K_{t-1}. At the end of period $t-1$, the financial intermediary stores the household's savings K_{t-1} with a

[5]Notice that one needs to derive the aggregate wage rate consistent with $w_t L_t = w_t^T L_t^T + w_t^N L_t^N$ where L_t is identified by the non-linear constraint in (2.6). This is: $w_t = [\chi(w_t^T)^{1+\epsilon} + (1-\chi)(w_t^N)^{1+\epsilon}]^{1/(1+\epsilon)}$

[6]Relaxing this assumption with another standard assumption such as monopolistically competitive intermediate goods producers, as common in a New-Keynesian setting, would not alter the results that are shown below as long as mark-ups of price over marginal costs in the two sectors are constant. Which is a standard assumption in New Keynesian models: with no price rigidity mark-ups are uniquely determined by the *constant* elasticity of demand for intermediate goods.

[7]Alternatively, one could easily assume that the households have direct ownership of physical capital and rent it to the firms. The no-arbitrage condition would hold in any equilibrium where households maximized utility.

costless technology. In period t this intermediary transforms savings into capital: $K_{t-1} = K_t^N + K_t^T$. How? Capital K_{t-1} is rented to the firms which pay rental rate $r_t^k K_t$ and return undepreciated capital $(1-\delta)K_t$ to the intermediary. This financial intermediary pays interest r_t to the household $(1+r_t)K_{t-1}$ making zero profit so that $r_t^k K_{t-1} + (1-\delta)K_{t-1} - (1+r_t)K_{t-1} = 0$. Hence the household's savings K_{t-1} give a return:

$$r_t = r_t^k - \delta \tag{2.13}$$

that is, the real interest is equal to the marginal product of capital (r_t^k) net of depreciation (δ).

2.5 Clearing

In addition to the labor market, equation (2.2), the capital market and the markets for the two goods clear:

$$K_{t-1} = K_t^N + K_t^T \tag{2.14}$$
$$C_t^N = Y_t^N \tag{2.15}$$
$$C_t^T + K_t = (1-\delta)K_{t-1} + Y_t^T \tag{2.16}$$

It is assumed that only T-goods can be used for the purpose of capital investment.

2.6 Shocks

Aggregate uncertainty is not considered in this model. Instead, the model has a perfect-foresight set-up: there is a one-time shock, that moves the system outside the initial steady state, where the time-path of all exogenous demographic variables is revealed; the initial shock is unanticipated but agents are perfectly aware of the entire path revealed, including the fact that at some point in the future demographic variables will remain constant forever, at the level reached at that point in the future. Essentially, the exogenous variation in the number of people $N_{t,j}$ in all period t and in all cohorts j is the only shock in the model. It gives rise to three exogenous variables, φ_t, L_t, ζ_t.

2.7 Equilibrium

Given the dynamics of the exogenous number of people $N_{t,j}$ in all period t and in all cohorts j (which leads to the exogenous dynamics of the parameters φ_t, L_t, ζ_t, according to (2.3) and (2.2) and by averaging in each period the age-dependent survival probabilities implied by (2.1)) and sectoral production technologies A_t^T, A_t^N, equilibrium for this (closed, perfectly competitive) economy is a sequence of prices $\{w_t^T, w_t^N, w_t, r_t, r_t^k\}_{t=0}^{\infty}$ and quantities $\{L_t^T, L_t^N, K_t^T, K_t^N, C_t^N, C_t^T, Y_t^N, Y_t^T\}_{t=0}^{\infty}$, such that:

1. the representative household solves (2.4), maximizing expected utility function subject to the budget constraint (2.5) and the preference to work in either sector (2.6);

2. the fiscal authority sets a tax rate (2.8) such that its budget is balanced in each period given a certain individual pension transfer (2.7);

3. the representative firms maximize profits solving (2.11) and (2.12) given production functions (2.9) and (2.10);

4. the financial intermediary ensures no arbitrage (2.13);

5. the markets for capital (2.14) and for goods (2.16), (2.15) clear.

3 Analytics

3.1 Characterization of the model

In order to simplify the analytic environment, some assumptions are made. First, to have wage equalization in all periods, i.e. $w_t^T = w_t^N = w_t$ for all t, it is assumed that there is perfect mobility of labor (i.e. $\epsilon \to \infty$). This implies $L_t^N + L_t^T = L_t$. Hence, with the notation above, a time-varying fraction χ_t of the exogenous labor supply L_t is employed in the T-sector. Consequently, a fraction $(1 - \chi_t)$ is employed in the N-sector. That is, χ_t is an endogenous variable.
Furthermore, in order to isolate the impact of demographic change, the analysis abstracts from exogenous technological change setting $A_t^T = A^T = 1$, $A_t^N = A^N = 1$ for all t. In Appendix B it is shown that under these assumptions, the equilibrium of the model can be fully characterized by four equations:

$$w_t = (1-\psi)\left(\frac{\psi}{r_t+\delta}\right)^{\frac{\psi}{1-\psi}} \tag{3.1}$$

$$\widetilde{K}_{t-1} = \frac{w_t\psi}{(r_t+\delta)(1-\psi)} \tag{3.2}$$

$$\chi_t = \gamma + (1-\gamma)\frac{[\widetilde{K}_t L_{t+1}^g - (1-\delta)\widetilde{K}_{t-1}]}{\widetilde{K}_{t-1}^{\psi}} \tag{3.3}$$

$$1 + r_{t+1} = \frac{(L_{t+1}^g)^{\sigma}}{\beta\zeta_{t+1}^g (N_{t+1}^g)^{\sigma}}\left(\frac{\widetilde{K}_t}{\widetilde{K}_{t-1}}\right)^{\psi\sigma}\left(\frac{1-\chi_{t+1}}{1-\chi_t}\right)^{\sigma} \tag{3.4}$$

where the four endogenous variables are $w_t, r_t, \chi_t, \widetilde{K}_t$ with $\widetilde{K}_{t-1} \equiv K_{t-1}/L_t$ and the three exogenous time-varying variables are $L_{t+1}^g \equiv L_{t+1}/L_t$, $N_{t+1}^g \equiv N_{t+1}/N_t$, $\zeta_{t+1}^g \equiv \zeta_{t+1}/\zeta_t$.

Equations (3.1) and (3.2) derive directly from the representative firms' optimal conditions in presence of wage equalization and are standard macro relations: the inverse relationship between the real wage and the real interest rate and how this relates to the capital-labor ratio. Perfect mobility of both labor ($L_t^T + L_t^N = L_t$ with $L_t^T = \chi_t L_t$) and capital ($K_t^T + K_t^N = K_{t-1}$) ensures that the capital-labor ratio is the same in the two sectors and equal to the aggregate capital-labor ratio: $K_t^T/L_t^T = K_t^N/L_t^N = w_t\psi/((r_t+\delta)(1-\psi)) = K_{t-1}/L_t \equiv \widetilde{K}_{t-1}$ for all t. That is, a fraction χ_t of aggregate capital is employed in the T-sector and, complementarily, a fraction $1-\chi_t$ of aggregate capital is employed in the N-sector: $K_t^T = \chi_t K_{t-1}$, $K_t^N = (1-\chi_t)K_{t-1}$. Notice that factors get their marginal product as remuneration: w_t for labor and $r_t^k = r_t + \delta$ for capital (no arbitrage condition (2.13) applies).

Equation (3.3) identifies the time-varying fraction χ_t of exogenous efficiency units of labor (L_t) employed in the T-sector. It is obtained by using the optimal conditions into the clearing condition for T-goods (2.16). It provides some intuitions concerning the sectoral dynamics of the model. First, the fraction of labor employed in the T-sector needs to be at least sufficient to meet the consumption demand of T-goods, i.e. χ_t needs to be at least as big as the fraction of consumption devoted to T-goods γ.[8] Second, χ_t is proportional to the investment-output ratio $(\widetilde{K}_t L^g_{t+1} - (1-\delta)\widetilde{K}_{t-1})/\widetilde{K}^{\psi}_{t-1}$. Indeed, production function (2.9) can be rewritten as $Y^T_t = (A^T_t)^{1-\psi}(K^T_t/L^T_t)^{\psi}\chi_t L_t = (A^T_t)^{1-\psi}(\widetilde{K}_{t-1})^{\psi}\chi_t L_t$. Symmetrically, in the N-sector: $Y^N_t = (A^N_t)^{1-\psi}(\widetilde{K}_{t-1})^{\psi}(1-\chi_t)L_t$. It follows that aggregate output (in terms of T-goods) is:

$$Y_t \equiv Y^T_t + Z_t Y^N_t = L_t \widetilde{K}^{\psi}_{t-1}\left[\chi_t (A^T_t)^{1-\psi} + (1-\chi_t) Z_t (A^N_t)^{1-\psi}\right] = L_t \widetilde{K}^{\psi}_{t-1}(A^T_t)^{1-\psi}$$

where the last equality follows from the fact that by optimal conditions the relative price of N-goods is equal to the relative productivity: $Z_t = (A^T_t/A^N_t)^{1-\psi}$. With the assumption $A^T_t = A^N_t = 1$ for all t, it results:

$$\widetilde{Y}_t \equiv \frac{Y_t}{L_t} = \widetilde{K}^{\psi}_{t-1} \tag{3.5}$$

Furthermore, from the law-of-motion of capital, which is inserted in the clearing condition (2.16), investment is $K_t - (1-\delta)K_{t-1}$ which can be easily rewritten as $\widetilde{K}_t L^g_{t+1} - (1-\delta)\widetilde{K}_{t-1}$ using $\widetilde{K}_{t-1} \equiv K_{t-1}/L_t$. Therefore, χ_t is time-varying as long as the investment-output ratio is time-varying, proportionally to the share of consumption devoted to N-goods. For example, if the exogenous demographic forces lead the investment-output ratio to increase, then the fraction of labor employed in the T-sector increases. The reason is that it is assumed that capital investment can be generated only with T-goods so that the economy needs to produce a sufficient amount of T-goods to meet the investment needs. To do so, a sufficient amount of labor input needs to be directed to the T-sector. Since N-goods can be produced only with capital generated via T-goods, there needs to be a movement of labor from the N-sector to the T-sector whenever a higher fraction of aggregate output goes to investment, to meet the capital demand in the N-sector which depends on the consumption demand of N-goods (as captured by the share of consumption devoted to N-goods $1-\gamma$). If capital investment was always zero, then output in the T-sector would always be equal to consumption of T-goods so that there would be no surplus of T-goods to be employed as capital in the N-sector with consequently no need to relocate some labor from the N-sector to the T-sector (χ_t would be constant at γ).
Another intuition that can be gained from (3.3) is that the fraction of labor employed in T-sector (χ_t) is positively related to the share of consumption devoted to T-goods (γ). That is, if there is an infatuation for goods of the T-sector which raises the consumption share of T-goods, then the fraction of labor employed in

[8] From the representative household's first order conditions, the share of consumption expenditure devoted to T-goods is:

$$\frac{C^T_t}{C^T_t + Z_t C^N_t} = \frac{\frac{\gamma}{1-\gamma} Z_t C^N_t}{\frac{\gamma}{1-\gamma} Z_t C^N_t + Z_t C^N_t} = \gamma$$

the T-sector needs to increase, everything else being equal.[9]

Finally, equation (3.4) describes the evolution of the real interest rate which is extensively discussed in section 3.3.

3.2 Steady states

Denote variables at the initial steady state with no time subscript. It is assumed that in the initial steady state there is no growth of the demographic variables. By (3.4) this implies that the real interest rate is equal to the inverse of the discount factor:

$$1 + r = \frac{1}{\beta} \tag{3.6}$$

Given r, the wage rate w is easily pinned down by (3.1). Then, the capital-labor ratio $\widetilde{K} \equiv K/L$ is pinned down by (3.2) and χ by (3.3). All the other variables are then easily identified analytically as functions of parameters (cf. Appendix B.2).

It is assumed that after year 2100 the three exogenous growth rates $(L^g_{t+1}, N^g_{t+1}, \zeta^g_{t+1})$ revert slowly to 1. Hence, the system (3.1)–(3.4) reverts to the initial steady state eventually. However, notice that in a balanced growth path characterized by constant growth rates of the exogenous variables ($N^g_t = N^g, L^g_t = L^g, \zeta^g_t = \zeta^g$ for all t), and so a constant capital-labor ratio, the constant real interest rate would be:

$$1 + r = \frac{(L^g)^\sigma}{\beta \zeta^g (N^g)^\sigma} \tag{3.7}$$

3.3 The impact of aging on the real interest rate: three channels

The system (3.1)–(3.4) can be further managed and simplified to get further intuitions. First notice that using (3.1) into (3.2) the capital-labor ratio is:

$$\widetilde{K}_{t-1} = \left(\frac{\psi}{r_t + \delta} \right)^{\frac{1}{1-\psi}} \tag{3.8}$$

Then, using $\widetilde{Y}_t = \widetilde{K}^\psi_{t-1}$ by (3.5), have investment (in unit of labor efficiency): $\widetilde{I}_t = \widetilde{K}_t L^g_{t+1} - (1-\delta)\widetilde{K}_{t-1}$, and define the investment-output ratio – which, by the assumption of closed-economy, is equal to the net

[9]Formally:

$$\frac{\partial \chi}{\partial \gamma} > 0 \Longleftrightarrow 1 - \frac{[\widetilde{K}_t L^g_{t+1} - (1-\delta)\widetilde{K}_{t-1}]}{\widetilde{K}^\psi_{t-1}} > 0$$

which is always verified as long as consumption is positive.

saving-output ratio or, shortly, *net saving rate*[10] – as:

$$\widetilde{\iota}_t \equiv \frac{\widetilde{I}_t}{\widetilde{Y}_t} = \frac{\widetilde{K}_t L^g_{t+1} - (1-\delta)\widetilde{K}_{t-1}}{\widetilde{K}^{\psi}_{t-1}} = \frac{I_t}{Y_t} \equiv \iota_t \tag{3.9}$$

Then, by (3.3):

$$1 - \chi_t = (1-\gamma)(1-\iota_t)$$

which plugged into (3.4) gives:

$$1 + r_{t+1} = \frac{(L^g_{t+1})^{\sigma}}{\beta \zeta^g_{t+1} (N^g_{t+1})^{\sigma}} \left[\left(\frac{\widetilde{Y}_{t+1}}{\widetilde{Y}_t} \right) \left(\frac{1-\iota_{t+1}}{1-\iota_t} \right) \right]^{\sigma} \tag{3.10}$$

with $\widetilde{Y}_t = \widetilde{K}^{\psi}_{t-1}$ and (3.8), the system is solved by (3.10): a unique recursive equation which identifies a path for the real interest rate for given paths of the exogenous variables (L^g_t, N^g_t, ζ^g_t for all t) given an initial and final value for the real interest rate.

Following Krueger and Ludwig (2007) and Carvalho, Ferrero, and Nechio (2016), there are three main channels through which population aging (here mainly captured by a decline in L^g_t/N^g_t, i.e. the growth rate of the effective labor-population ratio) impacts the real interest rate:

1. *Lower labor input* (capital demand). Labor as a production factor becomes scarcer so that *ceteris paribus* capital per worker rises depressing the marginal product of capital: inward shift of the capital demand, i.e. lower r_t. This is akin to a permanent slowdown in productivity growth;

2. *Higher life expectancy* (capital supply). Individuals expect to live longer so that *ceteris paribus*, depending on the benefits set in place by the pension scheme, they increase their savings in anticipation of a longer retirement period: outward shift of the capital supply, i.e. lower r_t. This is akin to a preference shock (individuals becoming more patient);

3. *Higher proportion of dissavers* (capital supply). The age composition of the population shifts towards relatively older individuals who are dissavers according to the life-cycle model: *ceteris paribus* inward shift of the capital supply, i.e. higher r_t. This is akin to a demand shock that pushes up aggregate consumption.

To better understand the three channels consider variables *per capita*, i.e. divide them by N_t. Notice that by accounting identity aggregate consumption must be:

$$C_t = Y_t - I_t$$

[10]By the law-of-motion of capital, the saving rate is $(K_t - K_{t-1})/Y_t = (I_t - \delta K_{t-1})/Y_t$. The net (of capital depreciation) saving rate is $(K_t - (1-\delta)K_{t-1})/Y_t = I_t/Y_t$

i.e. $1 - \iota_t = C_t/Y_t$. Define variables per-capita with a bar: $\overline{X}_t \equiv X_t/N_t$ for each variable X_t (because of its pre-determined nature, capital: $\overline{K}_t = K_t/N_{t+1}$). It is easy to show that the system to solve becomes:

$$\overline{C_t} = \overline{Y_t} - [\overline{K}_t N^g_{t+1} - (1-\delta)\overline{K}_{t-1}] \tag{3.11}$$

$$\overline{K}_{t-1} = \overline{L}_t \left(\frac{\psi}{r_t + \delta}\right)^{\frac{1}{1-\psi}} \tag{3.12}$$

$$\overline{Y}_t = \overline{K}^{\psi}_{t-1} \overline{L}^{1-\psi}_t \tag{3.13}$$

$$1 + r_{t+1} = \frac{1}{\beta \zeta^g_{t+1}} \left[\frac{\overline{C}_{t+1}}{\overline{C}_t}\right]^{\sigma} \tag{3.14}$$

Notice that the main exogenous variable is the effective labor-population ratio $\overline{L}_t = L_t/N_t$ and what matters for the determination of the real interest rate is its growth rate $\overline{L}^g_{t+1} \equiv (L_{t+1}/N_{t+1})/(L_t/N_t)$ which steadily decreases from the 80s till 2030 (cf. Figure 4). To understand the three channels above, consider the following three equivalent ways of writing the Euler equation (3.14):

$$1 + r_{t+1} = \frac{1}{\beta \zeta^g_{t+1}} \left[\frac{\overline{C}_{t+1}}{\overline{C}_t}\right]^{\sigma} = \frac{1}{\beta \zeta^g_{t+1}} \left[\frac{\overline{Y}_{t+1}}{\overline{Y}_t}\left(\frac{1-\iota_{t+1}}{1-\iota_t}\right)\right]^{\sigma} = \frac{1}{\beta \zeta^g_{t+1}} \left[(\overline{L}^g_{t+1})^{1-\psi}\left(\frac{\overline{K}_t}{\overline{K}_{t-1}}\right)^{\psi}\left(\frac{1-\iota_{t+1}}{1-\iota_t}\right)\right]^{\sigma}$$

with $1 - \iota_t = \overline{C}_t/\overline{Y}_t$.

Channel 1. A decrease of the growth rate of the labor-population ratio ($\overline{L}^g_{t+1}$) is akin to a negative shock for the growth rate of total factor productivity for output per capita growth:

$$\frac{\overline{Y}_{t+1}}{\overline{Y}_t} = (\overline{L}^g_{t+1})^{1-\psi}\left(\frac{\overline{K}_t}{\overline{K}_{t-1}}\right)^{\psi}$$

thus, a decrease of $\overline{L}^g_{t+1}$ leads firms to demand less capital (the demand for capital per capita is given by equation (3.12)). By (2.2) one can separate two drivers of labor input: (*i*) *labor quantity*, namely the raw number of people in the workforce (between age 15 and 64 in the baseline calibration); (*ii*) *labor efficiency*, namely the contribution to the effective scarcity of labor due to the fact that workers at different ages have a different productivity (cf. Figure 3).

Channel 2. The representative household realizes that the growth rate of the number of effective workers (L_t) in support of the number of total consumers (N_t) is shrinking. Therefore, with the goal of smoothing consumption per capita into the future (cf. Euler equation (3.14)), the representative household anticipates this change with a willingness of consuming less and saving more. A decrease of $\overline{L}^g_{t+1}$ enters the Euler equation as a *positive* shock to the discount factor (scaled by σ, i.e. the inverse of the intertemporal elasticity of substitution in consumption), meaning that the representative household becomes more patient (more so the higher the value of σ). An increase in the growth rate of the average survival probability in the economy (ζ^g_{t+1}) amplifies this mechanism by making the representative household even more patient.

Channel 3. As the effective labor force becomes scarcer, it is optimal to reduce the pace of capital accumulation in support of labor. Thus, the aggregate investment rate (ι_t), that by the assumption of closed economy is equal to the net saving rate (cf. footnote 10), decreases with aging leading *ceteris paribus* to an increase of the real interest rate. In this way the model captures that more and more effective scarcity of those who save (workers) compared to the universe of consumers (total population), i.e. shifts of the age distribution in favor of dissavers, leads to an increasingly smaller saving rate and so to an increase of the interest rate, everything else being equal. Notice that what matters for intertemporal choices is the growth rate of one minus the net saving rate.

It is important to notice that the dynamic equilibrium of the model can be such that with aging the saving rate decreases (channel 3) but the shifts in desired saving (channel 2) and desired investment (channel 1) are such the the real interest rate is on a declining path.

4 Quantitative analysis

The goal of the quantitative analysis is to study the transition dynamics of the macroeconomic system from an initial to a final steady state, where the unique exogenous driving process is the time-varying demographic structure. The focus is on the euro area composed by 12 countries (EA12 henceforth) modeled as a closed economy.[11] The initial steady state is assumed to be year 1950 and agents learn about the future demographic development at the beginning of the following year. The demographic structure varies over the period 1950-2100. After year 2100, when the provided demographic projections end, it is assumed that the three exogenous growth rates ($L^g_{t+1}, N^g_{t+1}, \zeta^g_{t+1}$) revert slowly to 1. In the baseline simulation the system attains the final steady state only around year 2500, a horizon sufficiently far to proxy infinity and to avoid influences on the transition period of interest which extends a couple of decades into the future.

4.1 Calibration

Each period corresponds to one year. Table 1 summarizes the values of the parameters in the baseline calibration. Given the simplifying assumptions, perfect labor mobility between sectors ($\varepsilon \to \infty$) and no technological change (with $A^T_t = A^N_t = 1$ for all t), there are only five structural parameters in the model that do not depend on demographics. The capital elasticity of output in the Cobb-Douglas production function, ψ, is set to 0.3 as in Gomes, Jacquinot, and Pisani (2012)'s model for the euro area (where they also have it equal in the two sectors, tradable and nontradable). The annual depreciation rate of capital δ is set to a value slightly bigger than 0.09 consistently with Gomes, Jacquinot, and Pisani (2012).[12] Given ψ and δ the individual discount factor β is solved endogenously in the initial steady state in order to target an average capital-output

[11] EA12 is composed by the following countries: Austria (AT), Belgium (BE), Finland (FI), France (FR), Germany (DE), Greece (EL), Ireland (IE), Italy (IT), Luxembourg (LU), Netherlands (NL), Portugal (PT), Spain (ES).

[12] Gomes, Jacquinot, and Pisani (2012) set the depreciation rate at the quarterly frequency to 0.025. At the annual frequency this implies a depreciation rate $\delta = 1 - (1 - .025)^4 = 0.0963$.

ratio K/Y of 2.76.[13] [14] Given $1 + r = 1/\beta$ it can be easily shown that the discount factor that satisfies a targeted value of the capital-output ratio K/Y in the initial steady state is $\beta = 1/(\psi/(K/Y) + 1 - \delta)$. It gives a value for the discount factor of 0.9822 which in turn implies a value for the real interest rate in the initial steady state of about 1.81%. The share of consumption expenditure devoted to T-goods, γ, is computed as the following average: $\gamma = (1/F)\sum_{t=1}^{F}\sum_{j=0}^{J}\alpha_j N_{t,j}/N_t$ where the age-specific shares of consumption devoted to T-goods α_j are taken from Giagheddu and Papetti (2018). This gives a $\gamma = .484$, meaning that on average slightly less than 50% of the consumption expenditure goes to T-goods. Finally, the constant relative risk aversion (CRRA) parameter σ, which is the inverse of the intertemporal elasticity of substitution, is set to the value of 2.5, in line with the literature (cf. section 4.3.2).

For what concerns demographics, individul labor supply in efficiency units, h_j, is interpolated using the data-points provided by Domeij and Floden (2006), cf. Figure 3. It is assumed that individuals enter the world as workers at age 15, all retiring at age 65 which corresespond to $jr + 1 = 51$ (this is why h_j drops abruptly to zero after age 64). An assumption which is standard, cf. Kara and von Thadden (2016), Bielecki, Brzoza-Brzezina, and Kolasa (2017) for the euro area.

Table 1: Baseline calibration: parameter values

Parameter	Value	Note
Structural:		
ψ	0.3	bias towards capital in the production function (Cobb-Douglas), cf. Gomes, Jacquinot, and Pisani (2012)
δ	0.0906	depreciation rate of capital, cf. Gomes, Jacquinot, and Pisani (2012)
β	0.9822	individual discount factor, endogenous in the initial steady state: $\beta = 1/(\psi/(K/Y) + 1 - \delta)$, $K/Y = 2.76$
γ	0.484	consumption share on T-goods, average based on age-consumption shares from Giagheddu and Papetti (2018)
σ	2.5	constant relative risk aversion (CRRA), cf. section 4.3.2
Demographic:		
J	86	terminal life-age (100). Death with certainty at age 101
jr	50	terminal working-age (64), cf. Kara and von Thadden (2016), Bielecki, Brzoza-Brzezina, and Kolasa (2017)
h_j	Figure 3	individual life-cycle labor supply in efficiency units (age 15–64). Source: Domeij and Floden (2006)
N_t^g	Figure 4	growth rate of population (age 15–100). Source: UN
L_t^g	Figure 4	growth rate of aggregate labor supply in efficiency units. Source: UN , cf. h_j
ζ_t^g	Figure 4	growth rate of average aggregate unconditional survival probability. Source: implied by UN, applying (2.1)
Simplifying:		
A^T	1	labor-technology level in the T-sector
A^N	1	labor-technology level in the N-sector
ε	∞	degree of sectoral substitutability of labor ($\epsilon \to \infty$: perfect mobility of labor)

The empirical number of people, $N_{t,j}$, by single age group $j \in \{15, 16, ..., 100+\}$ for each year t in the

[13] The average for EA12 is taken over the time-range available, 1970-2016, obtained by weighting each country with its real GDP share in year 2000. The series used are: "Gross capital formation (constant LCU)", "Gross fixed capital formation (constant LCU)" and "GDP (constant LCU)", data source: World Development Indicators (WDI) by the World Bank (update: January 2018). The capital stock is estimated by applying the perpetual inventory method (cf. Technical Appendix of Cardi and Restout (2015)). The initial capital stock (the base year is 1970, the first year data are available for all EA12 countries) is computed using the formula:

$$K_{1970} = \frac{I_{1970}}{g_I + \delta_K}$$

where I_{1970} corresponds to the gross capital formation in 1970. g_I is the average growth rate, while δ_K is set to 6% (cf. McQuinn and Whelan (2016)). The capital stock is obtained via the neoclassical law-of-motion: $K_{t+1} = (1 - \delta)K_t + I_t$.

[14] Notice that targeting a capital-output ratio $K/Y = 2.76$ with a depreciation rate $\delta = .0906$ gives an investment output ratio of about 0.25 in the initial steady state. Indeed, by the law-of-motion of capital in steady state, $\delta K = I$. That is, $I/Y = \delta(K/Y) = .0906 \times 2.76 = 0.25$.

time-range 1950-2100 is taken from the United Nations (UN, 2017) *World Population Prospects: The 2017 Revision*.[15] These data allow to identify immediately the growth rate of the population (people aged between 15 and 100), N_t^g, and the growth rate of labor in efficiency units L_t^g:

$$N_t^g = \frac{\sum_{j=0}^{J} N_{t,j}}{\sum_{j=0}^{J} N_{t-1,j}}, \quad L_t^g = \frac{\sum_{j=0}^{jr} h_j N_{t,j}}{\sum_{j=0}^{jr} h_j N_{t-1,j}} \tag{4.1}$$

The computation of the growth of the average unconditional survival probability ζ_t^g is less immediate. First, the individual unconditional survival probabilities $\pi_{t,j}$ are retrieved in the data by applying formula (2.1) which implies the following evolution of the size of each cohort: $N_{t+j,j} = \pi_{t+j,j} N_{t,0}$.[16] Then, the average aggregate unconditional survival probability across cohorts j in each year t is computed by summing each $\pi_{t,j}$ weighted by the share of people in the economy of age j at time t, $N_{t,j}/N_t$, so that the growth rate of interest is:

$$\zeta_t^g = \frac{\sum_{j=0}^{J} \pi_{t,j}(N_{t,j}/N_t)}{\sum_{j=0}^{J} \pi_{t-1,j}(N_{t-1,j}/N_{t-1})} \tag{4.2}$$

The three exogenous variables N_t^g, L_t^g, ζ_t^g tracking the evolution of the age-structure over time, as well as the growth rate of the "working age population ratio" ($L_t^g - N_t^g$), are plotted in Figure 4.[17] Population (N_t^g) and labor (L_t^g) are growing approximately at the same rate before 1990. In that year their growth rate is more than 0.6% per year. However, in the subsequent periods, there is an increasing divergence: in year 2035 the population growth rate is about null while that of labor is negative at about -0.7%. Figure 4 plots also the growth rate of the "working age population ratio" i.e. the difference between the growth rate of labor and population ($L_t^g - N_t^g$). As discussed earlier in section 3.3, the growth rate of the working age population ratio is the main exogenous variable. Its shape, as it will be clear in the next section, is close to the shape of the real interest rate over the transition. Finally, the growth of the average survival probability (ζ_t^g) is always positive over the period considered, but its dynamics is pretty flat. Therefore, one can expect only a level-effect for this part of the life-expectancy channel on the real interest rate.

[15] Before year 1990, the number of people aged more than 80 are grouped together in the set 80+ for all countries. Therefore, as a strategy to identify the number of people in each single age group after age 80 for years 1950-1989, the implied survival probabilities of 1990 for those aged more than 80 have been applied backwards.

[16] Consider the number of people entering the world at time t: $N_{t,0}$. According to (2.1) the number of survivors next period is $N_{t+1,1} = s_{t+1,1} N_{t,0}$. By the same formula, two years after the survivors will be: $N_{t+2,2} = s_{t+2,2} N_{t+1,1} = s_{t+2,2} s_{t+1,1} N_{t,0} = \pi_{t+2,2} N_{t,0}$. Hence, generally, $N_{t+j,j} = \pi_{t+j,j} N_{t,0}$. Notice that data from United Nations are available since 1950. Before 1950 it is assumed that the system is in steady state. Hence, in the data it is assumed that in all periods before 1950 the demographic structure of 1950 applies. For example, a person born in 1946 (i.e. aged 4 in 1950) will have the same unconditional probability of turning 5 in 1951 of a person born in 1950 (that will turn 5 in 1955).

[17] The values of the three variables have been conveniently smoothed using a "loess" method (local regression using weighted linear least squares and a 2nd degree polynomial model) of the smooth function in Matlab with a span of .25, a low value to preserve the actual data but at the same time avoid kinks.

4.2 Main results

4.2.1 Baseline

The continuous line in Figure 5 shows that in response to the exogenous demographic transition in the model the real interest rate decreases by about 1 percentage point between 1990 and 2030. The model seems to capture also some upward pressure before 1990 found in the data. Because of no arbitrage, the risk-free real interest rate is equal to the marginal product of capital net of depreciation (net MPK), cf. equation (2.13). Coherently, the series generated by the model is compared with an empirical measure of the MPK (note of Figure 5 describes how it is constructed). Visually, the model (which has demographic change as unique driver) captures some low-frequency movement of the real interest rate in the data and predicts that it will remain low in the projected horizon. The dashed line shows the evolution of the exogenous component in the equilibrium equation for the real interest rate (3.4), i.e. the working-age population ratio corrected for the risk-aversion parameter σ, the discount rate β and the average growth of the survival probability ζ_t^g: $(L_t^g)^\sigma/(\beta\zeta_t^g(N_t^g)^\sigma)$. The difference between the continuous line and the dashed-line is due to the endogenous effects in the model which smooth the exogenous impact.

Figure 5 evaluates also what happens when ζ_t^g is set equal to 1 for all t ("no survival risk"). This is the case that emerges when the aggregate representation of the OLG model is derived (cf. Appendix A). It can be seen (comparing the continuous line with the dashed-dotted line) that the change of the average survival probability has a downward impact on the equilibrium real interest rate, but it is only a relatively small level effect. Between 1990 and 2030, approximately the peak to trough for the simulated real interest rate, the model predicts a decrease of the real interest of about 1 percentage point under the "baseline" specification, 1.25 percentage points under the "no survival risk" specification.

4.2.2 Comparison with empirics

Considering the specification with $\zeta_t^g = 1$ for all t, Figure 7a shows the demeaned series over the period 1960–2030 decomposing the demographic impact in two components: stemming from the change in the mere number of people in the working-age ("labor quantity", number of people aged 15–64), i.e. using $L_t = \sum_{j=0}^{jr} N_{t,j}$, and from their age-varying productivity ("labor efficiency"), i.e. using $L_t = \sum_{j=0}^{jr} h_j N_{t,j}$. It shows that demographics predicts a rise (in the 70s) and then a fall (after a peak around 1990) of the natural real interest rate. The rise is explained by a favorable development of both the share of people in the working age and the fact that they belong to a part of the age-distribution where, according to the productivity-profile h_j (cf. Figure 3), productivity is relatively higher. Approximately around 1990 the growth rate of the working-age population ratio starts to decrease (cf. Figure 4) and so does the contribution of "labor quantity" for the level of the real interest rate. Compared to the mean over the period 1960-2030, its contribution turns negative slightly after 2000. However, the real interest rate turns negative (relative to the mean) later. The reason is that the people remaining in the labor force in this period are still sufficiently productive to provide an upward pressure on the real interest rate ("labor efficiency" bars). The inflection point is around 2010, when the impact of both components turns negative. While the main impact is due to the fact that there are less people in the labor force, a non-negligible factor is that those people in the labor force are older than before, i.e. less productive, thus contributing to the dampening effect.

Under the specification used in Figure 7a the unique exogenous drivers are L_t^g and N_t^g and what enters the Euler equation, cf. (3.4), is $(L_{t+1}^g)^\sigma/(N_{t+1}^g)^\sigma$, i.e. the expected growth rate of the labor-population ratio scaled by the risk-aversion parameter σ. In the econometric literature it is common to use the population share of people in the working-age, or in different age-bins, as a variable capturing the demographic dynamics. Figure 7b shows the estimate of the natural real interest rate provided by Fiorentini, Galesi, Pérez-Quirós, and Sentana (2018) from a panel error correction model (ECM) at annual frequency over the period 1899-2016 for an unbalanced panel of 17 advanced economies. Demographics captured by the share of people in the young age (20–39) turn out to be the main driver of the natural rate, in spite of the presence of TFP growth and a measure of risk from interest rates spreads. It is striking that the resulting shape over time of the natural real interest rate from this econometric estimate is very close to the one obtained from the aggregate representation of the OLG model, even if the magnitude tends to be bigger in the econometric model.

4.2.3 Identifying the channels

Following the explanatory channels in section 3.3, the decrease of the growth rate of the labor population-ratio (L_t^g/N_t^g) per se tends to put downward pressure to the real interest rate, both by leading to scarcer labor input (less desired demand for capital by firms, channel 1) and by making the representative household more patient in the expectation of having to smooth per capita consumption with a population that decreases by less than the number of people providing labor for production (more desired supply of capital by households, channel 2). To what extent these shifts in preferences are mitigated by the fact that the investment rate in capital (i.e. the net saving rate by the assumption of closed-economy, cf. footnote 10) shrinks over time (because it is optimal to do so to complement in production a shrinking labor force)?[18] In other terms, is channel 3 highlighted in section 3.3 a relevant mitigating factor of the overall dampening effect of demographics on real interest rate seen above? To answer this question, a model where the investment rate is forced to be constant in each period (at the initial steady state value) is compared to the baseline specification. In particular it is assumed that the investment rate is fixed at its initial value $\iota_t = \iota = 0.25$ for all periods t. Considering equations (3.9) and (3.10), the equilibrium real interest rate in this case is:

$$1 + r_{t+1} \quad = \quad \frac{(L_{t+1}^g)^\sigma}{\beta\zeta_{t+1}^g(N_{t+1}^g)^\sigma}\left(\frac{\widetilde{K}_t}{\widetilde{K}_{t-1}}\right)^{\psi\sigma}$$

where the capital-labor ratio $\widetilde{K}_t$ is forced to be such that $(\widetilde{K}_t L_{t+1}^g - (1-\delta)\widetilde{K}_{t-1})/\widetilde{K}_{t-1}^\psi = \iota$. Figure 6 shows that if the investment rate was forced to be fixed in each period, then the real interest rate would have been higher in the 80s and it would have decreased more after 1990. The shaded are in the figure quantifies the impact of channel 3: after 2000 the age-distribution of the population shifts towards cohorts that imply a lower net saving rate, as a result the real interest rate from the full model is always at a higher level than

[18] In the simulation with $\zeta_t^g = 1$ for all t the investment rate decreases steadily going from about 27% in 1990 to 24.5% in 2050.

in the model with fixed investment rate. At the maximum, around year 2025, the model predicts that this channel has an upward pressure on the real interest rate of about 0.4 percentage points. Overall, this channel is not so big quantitatively. If the investment rate was constant, from 1990 to 2030 the real interest rate would have decreased about 1.85 percentage points instead of the 1.25 percentage points of the compared case (the comparison is done for the "no survival risk" case, i.e. with $\zeta_t^g = 1$ for all t). Therefore, over the period 1990-2030, channel 1 and 2 are estimated to have a joint downward pressure on the real interest rate of about 1.85 percentage points, while channel 3 an upward pressure of about 0.6 (=1.85-1.25) percentage points.
Can the impact of channel 1 and 2 be disentangled as well? The continuous line in Figure 6 (the one that nets out the impact of channel 3) can be thought as resulting from two forces: one purely due to the fact that the number of elderly in the economy (those aged more than 64, as 65 is the first age of exogenous retirement) is varying over time for given fixed number of people in the working age; the other (residual) force due to the fact that the number of people in the working age (which interacts with the age-varying labor productivity) varies over time. Figure 8 does the exercise of disentangling the impact of channel 2 by showing the result of a simulation where the unique exogenous driver is the growth rate of a hypothetical population where the growth rate of the number of people in the working age is null, so that all the variation in the population growth rate is due to the change in the number of elderly in the economy. In this case, what enters Euler equation (3.4) is the reciprocal of this hypothetical population growth rate: it captures how much the representative household becomes more patient due to the fact that the aggregate life expectancy, given by the number of elderly in the economy, is increasing. By construction, the residual part, namely the difference between the series generated by fixing the investment rate and the series generated by the hypothetical population growth rate, is the contribution of the growth rate of the (effective) labor force. Figure 8 shows that, compared to the mean over the period 1960-2030, the inflection point around 2010 (highlighted also in Figure 7a, when the real interest rate turns on the negative side) is when all the channels start having effect with the expected sign according to the narrative exposed in section 3.3. It is notable that the unique channel that would allow an upward pressure to the real interest rate is not strong enough to prevent a decrease of the real interest rate. In the 80s and 90s these channels were acting in the opposite direction contributing to the rise of the real interest rate: again, channel 3 was not strong enough to counterbalance the other two channels. Overall, channel 1 and 2 seem to be more important in explaining the evolution of the real interest rate explained by demographics alone, contributing roughly evenly in the same direction.

4.3 Sensitivity

4.3.1 Capital-labor substitutability

In the previous section, it has been seen that a driver of the declining real interest rate over the demographic transition after 1990 is the scarcity of labor as production input. As labor becomes scarcer, the demand for capital by firms is reduced thus leading to a decrease of the real interest rate everything else being equal (cf. channel 2 in section 3.3). A criticism to the quantitative estimates above based on this channel is that the demand for capital comes from the assumption of a Cobb-Douglas production function (cf. (2.9), (2.10)) which has a unitary elasticity of substitution between labor and capital. This channel might not matter at all if one admits that capital and labor are sufficiently substitutable between each other. In the limit, when capital and labor are perfect substitutes, changes in relative factor quantities have no impact on relative factor prices. To what extent can a higher substitutability between labor and capital mitigate the negative impact on the real interest rate induced by the demographic transition? This question can be answered by introducing a constant elasticity of substitution (CES) production function. Assume that in each sector $s \in \{T, N\}$ output (Y_t^s) is produced according to this function:

$$Y_t^s \;=\; \left[\psi (K_t^s)^{\frac{\rho-1}{\rho}} + (1-\psi)(L_t^s)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}} \tag{4.3}$$

where ψ is, as in the baseline analysis above, the bias towards capital and ρ is the elasticity of substitution between labor (L_t^s) and capital (K_t^s). When $\rho \to 1$, the production function is again a Cobb-Douglas. When $\rho > 1$, capital and labor are said to be *gross substitutes*. In this case, a lower supply of one input leads to added demand for the other input. The opposite occurs when $\rho \leq 1$, in which case capital and labor are said to be *gross complements*. Therefore, when the labor supply decreases in the process of population aging with labor and capital as gross substitutes in production, the capital demand does not increase any longer as in the case of a Cobb-Douglas production function.

When the production function in both sectors is (4.3), the dynamic equilibrium is characterized by the following set of equations (derivation in Appendix B.3):

$$w_t \;=\; (1-\psi)(\widetilde{Y}_t)^{\frac{1}{\rho}} \tag{4.4}$$

$$\widetilde{K}_{t-1} \;=\; \left(\frac{w_t \psi}{(r_t+\delta)(1-\psi)}\right)^{\rho} \tag{4.5}$$

$$\chi_t \;=\; \gamma + (1-\gamma)\frac{\widetilde{K}_t L_{t+1}^g - (1-\delta)\widetilde{K}_{t-1}}{\widetilde{Y}_t} \tag{4.6}$$

$$r_{t+1} \;=\; \frac{(L_{t+1}^g)^{\sigma}}{\beta \zeta_{t+1}^g (N_{t+1}^g)^{\sigma}}\left(\frac{(1-\chi_{t+1})\widetilde{Y}_{t+1}}{(1-\chi_t)\widetilde{Y}_t}\right)^{\sigma} - 1 \tag{4.7}$$

$$\widetilde{Y}_t \;=\; \left[\psi (\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}} + (1-\psi)\right]^{\frac{\rho}{\rho-1}} \tag{4.8}$$

where the notation is the same one used in section 3.1. It is easy to see that when $\rho \to 1$, the system (4.4) – (4.8) tends to the system (3.1) – (3.5). The definition of elasticity of substitution can be grasped from equation (4.5): when relative factor prices ($w_t/(r_t+\delta)$) increase by 1%, the capital-labor ratio ($\widetilde{K}_{t-1}$) increases by $\rho\%$.

In his summary of the empirical literature, Chirinko (2008) concludes that "the weight of the evidence suggests that ρ lies in the range between 0.40 and 0.60".[19] Nonetheless, it might be interesting to consider values of ρ grater than unity. One could speculate that ongoing economic processes are changing the nature of capital, so that estimates based on historical data might not be reliable any longer. For example, automation

[19] Of the 31 sources listed, 26 shows a ρ strictly less than one with a median of 0.52. The maximum value, reported only by one source, points to a value of 2. The remaining sources point to $1 < \rho < 1.5$ with one exception where ρ is slightly bigger than 1.5.

can be thought as a process that by making labor increasingly superfluous in production is leading to an increasing degree of substitutability between labor and capital. Furthermore, there might be theoretical reasons to believe that $\rho > 1$. Karabarbounis and Neiman (2014) need ρ greater than one to explain the simultaneous decrease of the relative price of investment goods and the labor share of income. They estimate $\rho = 1.25$. Piketty (2014) too needs ρ greater than one to explain the fact that historically the capital share of income was lower when the capital-output ratio was lower, even though he does not provide estimates of ρ.[20]

Figure 9 plots the real interest rate resulting from the demographic transition for a range of sensible values of ρ based on Chirinko (2008). As expected, compared to the baseline ($\rho = 1$: Cobb-Douglas production function), as the most dramatic phase of the demographic change begins (around 2010, the inflection point according to the analysis done on Figure 7a), the demographic impact on the real interest rate is more negative for values of ρ smaller than 1, less negative for values bigger than 1. The numbers underlying these series are reported in Figure 10 as percentage points change between 1990 and 2030, a time range which roughly represents the peak to trough across the different specification. According to the range of values suggested by the empirical literature, $\rho \in [0.4, 0.6]$, the demographic impact on the real interest rate between 1990 and 2030 stands into the percentage points range: $[-1.7, -1.44]$. Using a Cobb-Douglas production function one gets a smaller negative impact: about -1 percentage points. For values of $\rho > 1$ the demographic impact is limited but still present. With the highest elasticity reported, $\rho = 1.6$, the negative impact is -0.66 percentage points.

Overall, despite higher substitutability between labor and capital dampens the baseline negative impact of the demographic transition on the real interest rate, the sign is not reversed for any sensible value of ρ. In fact, for values of ρ in the range reported by the empirical literature the demographic impact on the real interest rate is even more negative than what estimated in the baseline specification above, in the period between 1990 and 2030.

4.3.2 Intertemporal elasticity of substitution in consumption

Eichenbaum, Hansen, and Singleton (1988) suggest that an appropriate range for the constant relative risk aversion (CRRA) coefficient is $\sigma \in [0.5, 3]$, a common range in the business-cycle literature. Rarely $\sigma < 1$. Most often σ is simply set to 1, which corresponds to the case of logarithmic preferences, as it is easy to handle algebraically. Hall (1988) is often quoted to set σ to a value close to 5. In a systematic analysis of the literature on the estimates of the intertemporal elasticity of substitution in consumption (the inverse of σ), Havranek (2013) documents that "the typical range of calibrations lies between 0.2 and 2", corresponding to $\sigma \in [0.5, 5]$. This is the range of values considered in the sensitivity analysis depicted in Figures 11 and 12, despite the value of σ remains controversial.[21]

Figure 11 shows the transition paths of the real interest rate under different values of σ. It is apparent that as

[20] Cf. Rognlie (2014) for an assessment of Piketty (2014) in relation to ρ

[21] Havranek (2013) finds "strong publication bias: researchers report negative and insignificant estimates less often than they should, which pulls the mean estimate [of the inverse of σ] up by about 0.5", pointing that the "corrected mean of micro estimates ... is around 0.3–0.4", i.e. a mean value of σ between 2.5 and 3.34.

the intertemporal elasticity of substitution in consumption decreases (i.e. σ increases) demographic change exerts more downward pressure on the real interest rate, especially after 1990. Figure 12 documents the percentage points change of the real interest rate between 1990 and 2030 as predicted by the model (i.e. the numbers underlying Figure 11). Under logarithmic preferences over consumption for the representative household ($\sigma = 1$), the model predicts a change of only -0.41 percentage points. As σ increases the predicted change is more negative: -1 percentage points under the baseline calibration ($\sigma = 2.5$), till -1.5 percentage points when $\sigma = 5$. Notice from Figure 11 that for a sufficiently high value of σ, roughly higher than 4, the real interest rate turns in negative territory over the projected period.

4.3.3 Endogenous labor supply and different retirement ages

Model. The analysis above is done with exogenous labor supply in efficiency units, $L_t = \sum_{j=0}^{jr} h_j N_{t,j}$. Under the assumption of separable utility, the analysis can be extended to endogenous labor supply. In Appendix A it is shown that when individuals are allowed to choose endogenously how much to work (in the unitary set of hours available), the representative household's maximization problem to solve in order to approximate the OLG equilibrium is the following:

$$\begin{aligned} \max_{\{\widetilde{C}_t^T, \widetilde{C}_t^N, \widetilde{l}_t, \widetilde{K}_t\}} \quad & \sum_{t=0}^{\infty} \beta^t \left\{ \frac{N_t^\sigma}{L_t^{\sigma-1}} \frac{[(\widetilde{C}_t^T)^\gamma (\widetilde{C}_t^N)^{1-\gamma}]^{1-\sigma}}{1-\sigma} - \widetilde{\nu}_t \frac{(\widetilde{l}_t)^{1+\phi}}{1+\phi} \right\} \\ \text{s.t.} \quad & \widetilde{C}_t^T + Z_t \widetilde{C}_t^N + \widetilde{K}_t L_{t+1}^g = (1-\tau_t) w_t \widetilde{l}_t + (1+r_t)\widetilde{K}_{t-1} + \widetilde{T}_t \end{aligned}$$

where, as in the previous sections, choice variables are evaluated in units of labor efficiency. Particularly, $0 < \widetilde{l}_t < 1$ denotes the endogenous fraction of aggregate efficiency units of labor L_t chosen by the representative household. The parameter ϕ represents the inverse Frish elasticity of labor supply. The additional exogenous wedge $\widetilde{\nu}_t$ is attached to the aggregate disutility of labor and has the following expression:

$$\widetilde{\nu}_t \equiv \left[\sum_j N_{t,j} (\widetilde{h}_j)^{1+\frac{1}{\phi}} (\nu_j \lambda_j)^{-\frac{1}{\phi}} \right]^{-\phi}$$

where $\widetilde{h}_j = h_j / L_t$ and, as before, it is assumed that the welfare weights λ_j are the same across individuals, equal to 1. To identify this wedge one needs to parametrize the individual age-dependent disutility of labor ν_j – other than having data on cohort sizes $N_{t,j}$ and on age-varying productivity h_j.
The representative household chooses also how much to work in each sector $\widetilde{l}_t^T, \widetilde{l}_t^N$. Under the assumption of perfect mobility of labor ($\widetilde{l}_t = \widetilde{l}_t^T + \widetilde{l}_t^N$), using the same notation as above, a time-varying fraction χ_t of the endogenously chosen aggregate labor supply $\widetilde{l}_t$ is allocated to the T-sector. That is, $\widetilde{l}_t^T = \chi_t \widetilde{l}_t$, $\widetilde{l}_t^N = (1-\chi_t)\widetilde{l}_t$.

As labor supply is an endogenous choice, contrary to the setting with exogenous labor supply, the government matters by setting the labor income tax rate τ_t running the PAYGO pension system. Therefore, equations (2.7) and (2.8) matter for the determination of the equilibrium allocation. Revised to account for endogenous labor they read:[22]

$$
\begin{aligned}
d_t &= \bar{d} w_t (1-\tau_t)\bar{h} \\
\tau_t w_t \widetilde{l}_t &= d_t \widetilde{\Omega}_t = \widetilde{T}_t
\end{aligned}
$$

where $\widetilde{\Omega}_t = \sum_{j=jp}^{J} N_{t,j}/L_t$ is the effective *old-dependency ratio* in the model: the number of people to support with pension transfers over the number of effective workers. The age of pension transfers jp is allowed to differ from the age of full retirement $jr+1$.

On the firms' side, under the assumption of no exogenous technology parameters, the Cobb-Douglas production function in both sectors is rewritten taking into account the endogenous labor supply:

$$
\widetilde{Y}_t^T = (\widetilde{K}_t^T)^{\psi}(\widetilde{l}_t^T)^{1-\psi}, \quad \widetilde{Y}_t^N = (\widetilde{K}_t^N)^{\psi}(\widetilde{l}_t^N)^{1-\psi}
$$

Under the assumption of perfect competition and no arbitrage (cf. equation (2.13)) firms in each sector solve:

$$
\max_{\widetilde{K}_t^T, \widetilde{l}_t^T} \widetilde{Y}_t^T - w_t \widetilde{l}_t^T - (r_t+\delta)\widetilde{K}_t^T, \quad \max_{\widetilde{K}_t^N, \widetilde{l}_t^N} Z_t \widetilde{Y}_t^N - w_t \widetilde{l}_t^N - (r_t+\delta)\widetilde{K}_t^N
$$

Finally, clearing conditions (2.14), (2.15), (2.16) hold:

$$
\begin{aligned}
\widetilde{K}_{t-1} &= \widetilde{K}_t^T + \widetilde{K}_t^N \\
\widetilde{Y}_t^N &= \widetilde{C}_t^N \\
\widetilde{Y}_t^T &= \widetilde{C}_t^T + L_{t+1}^g \widetilde{K}_t - (1-\delta)\widetilde{K}_{t-1}
\end{aligned}
$$

Optimal conditions. In Appendix B.4 it is shown that the model with endogenous labor supply is charac-

[22]In this case the net replacement rate $\bar{d}$ is defined as the percentage devoted to pension benefits of (net of taxes) earnings measured as average over the full endowment of efficiency units of labor, h_j for $j = 0, 1, \cdots, jr$.

terized by the following optimal conditions:

$$
\begin{aligned}
1+r_{t+1} &= \frac{(L^g_{t+1})^\sigma}{\beta\zeta^g_{t+1}(N^g_{t+1})^\sigma}\left(\frac{(1-\iota_{t+1})\widetilde{Y}_{t+1}}{(1-\iota_t)\widetilde{Y}_t}\right)^\sigma \\
\widetilde{l}_t &= \left[\frac{N_t^\sigma}{\widetilde{\nu}_t L_t^{\sigma-1}}\gamma\left(\frac{\gamma}{1-\gamma}\right)^{\gamma(1-\sigma)-1}\left((1-\gamma)(1-\iota_t)\widetilde{Y}_t\right)^{-\sigma}(1-\tau_t)w_t\right]^{\frac{1}{\phi}} \\
1-\tau_t &= \frac{\widetilde{l}_t}{\widetilde{l}_t+\overline{dh}\widetilde{\Omega}_t} \\
\iota_t &= \frac{(\widetilde{K}_t L^g_{t+1}-(1-\delta)\widetilde{K}_{t-1})}{\widetilde{Y}_t} \\
\widetilde{Y}_t &= \widetilde{K}^\psi_{t-1}\widetilde{l}_t^{1-\psi} \\
w_t &= (1-\psi)\left(\frac{\widetilde{K}_{t-1}}{\widetilde{l}_t}\right)^\psi \\
\widetilde{K}_{t-1} &= \widetilde{l}_t\left(\frac{\psi}{r_t+\delta}\right)^{\frac{1}{1-\psi}}
\end{aligned}
$$

where the endogenous variables are $\{r_t, w_t, \widetilde{Y}_t, \widetilde{l}_t, \widetilde{K}_t, \tau_t, \iota_t\}_{t=0}^\infty$ and the exogenous variables (depending uniquely on demographics) for all periods t are:

$$
N_t=\sum_{j=0}^{J}N_{t,j},\quad L_t=\sum_{j=0}^{jr}h_jN_{t,j},\quad \zeta_t=\sum_{j=0}^{J}\pi_{t,j}\left(\frac{N_{t,j}}{N_t}\right),\quad \widetilde{\Omega}_t=\sum_{j=jr+1}^{J}\frac{N_{t,j}}{L_t},\quad \widetilde{\nu}_t=\left[\sum_j N_{t,j}(\widetilde{h}_j)^{1+\frac{1}{\phi}}(\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi}
$$

with $N^g_{t+1}=L_{t+1}/L_t, L^g_{t+1}=L_{t+1}/L_t, \zeta^g_{t+1}=\zeta_{t+1}/\zeta_t$. Compared to the case of exogenous labor supply, as exogenous variables there are also the old-dependency ratio $\widetilde{\Omega}_t$ and the wedge to the aggregate disutility of labor $\widetilde{\nu}_t$.

Calibration. The inverse Frish elasticity of labor supply is assumed to have the standard vale of $\phi = 2$ (cf. Chetty (2012)). Following Mariano, Christopher, and Kathryn (2010) and Jones (2018) the individual age-varying disutility of labor ν_j is assumed to be a cumulative density function of a normal distribution scaled in a way to ensure that endogenous labor $\widetilde{l}_t$ never exceeds one in all periods of the transition returning a reasonable participation rate in the initial steady state (cf. below). Figure 15 shows the shape and details the functional form and parameter values. It implies that as the population age-distribution shifts towards older cohorts the aggregate disutility of supplying labor increases ($\widetilde{\nu}_t$ increases) for a given level of exogenous aggregate labor supply. It is assumed that people start receiving pension transfers at age 65, corresponding to $j = jp = 51$. Contrary to the analysis with exogenous labor supply above, the age of full retirement (i.e. the age j after which h_j is always equal to zero) is extended from age 65 to age 80 (corresponding to age $j = jr + 1 = 66$).[23] In other terms, the aggregate endowment of efficiency units available in the economy L_t now encompasses people aged between 15 and 79 (before it was only between 15 and 65) but the

[23]Notice that when the retirement age is changed, the productivity profile h_j needs to be re-scaled in order to account for the individual productivity compared to the average, cf. Figure 3.

representative household can choose not to employ fully this labor endowment by choosing only a fraction $\widetilde{l}_t$ of it in each period. For the PAYGO pension system two scenarios are analyzed: (a) *constant replacement rate*: consistently with the notation above, the replacement rate is held constant at $\overline{d} = 0.45$ (value in line with Kara and von Thadden (2016) for the euro area) while the contribution rate τ_t is adjusted such that the government budget is balanced in each period (b) *endogenous replacement rate*: the contribution rate is fixed at the initial steady state level of $\tau_t = 0.0815$ (corresponding to a replacement rate of 45% in the initial steady state) while the replacement rate $\overline{d}$ is adjusted such that the government budget is balanced in each period. Finally, the initial steady state is such that the real interest rate has the same value assumed in the previous sections which means that the discount factor β has the same value as before. Then, one needs to solve for the implied labor choice $\widetilde{l}$ in the initial steady state. Given the parameterization of the aggregate disutility of labor it results that $\widetilde{l}$ is about 0.76 in the initial steady state, a value in line with calibration for the aggregate participation rate (cf. Kara and von Thadden (2016)).

Results. Figure 13 compares the path of the real interest rate under exogenous labor supply (for different retirement ages) with the case of endogenous labor supply under the two scenarios explained above, constant vs endogenous replacement rate. There are four main effects taken into account in Figure 13.

(a) *Increase in retirement age*. The dash-dotted line in Figure 13 represents the case with exogenous labor supply when people work till age 79 instead of 64 (baseline) and then they fully retire. The immediate effect of this measure is that labor becomes less scarce leading to a smaller decrease of the labor-population growth rate (cf. Figure 14). Therefore, the impact of demographic change on the real interest rate is less dramatic: between 1990 and 2030 it decreases about 0.75 percentage points (instead of 1 percentage point in the baseline). That is, increasing the retirement age of 15 years (age 80 instead of 65) mitigates the decrease of only 0.25 percentage points between 1990 and 2030. The reason is that the reform allows to supply more labor by a category of people that are relatively less productive than the average according to the productivity profile h_j. In this regard, *decreasing* the retirement age by the same amount of years (15) would lead to a much lower real interest rate (i.e. an additional decrease of more than 25 percentage points). There is no symmetric effect of a change in the retirement age because people at different age have a different level of productivity. However, since productivity decreases smoothly after age 65 (cf. Figure 3), any reform increasing the retirement age between age 65 and 80 would generate a higher path of the real interest rate in the period considered, lying between the continuous line and the dash-dotted line in Figure 13.

(b) *Endogenous labor supply*. If on top of increasing the retirement age one allows for endogenous labor supply, the first thing to notice is that the growth rate of the aggregate labor supply ($L_t\widetilde{l}_t$), hence the growth rate of the labor-population ratio ($L_t\widetilde{l}_t/N_t$), becomes more volatile largely driven by fluctuations in the growth rate of the wedge attached to the aggregate disutility of labor ($\widetilde{\nu}_t$). Then, what matters for the determination of factor prices is not the level of the labor supply but its growth rate. Therefore, despite with an endogenous choice of labor the aggregate labor supply is always smaller in level than in the case of exogenous labor supply with the same retirement age (because $0 < \widetilde{l}_t < 1$), the path of the growth rate of the labor-population ratio can be at a higher level and so can be the associated paths of the real interest rate (cf. dashed and dotted lines in Figures 13 and 14). Finally, the main source of variation in the choice of labor supply in this model, other than variations in $\widetilde{\nu}_t$, comes from how the tax rate is handled by the government running a pension system that gives balanced budget in each period. For this reason two types of government policy are analyzed next, with endogenous labor supply: holding the replacement rate fixed (and thus varying the tax rate) and holding the tax rate fixed (and thus varying the replacement rate). Since the effective old-dependency ratio $\widetilde{\Omega}_t$ increases throughout the whole transition period (going from 8.9% in 1950 to 32% in 2050), fixing the replacement rate means increasing the tax rate while fixing the tax rate means decreasing the replacement rate to keep the government budget balanced in each period in both cases. On the one hand, with a fixed replacement rate at 45% the tax rate in the model's simulation is predicted to steadily increase from a value of about 8% in 1950 to about 23% in 2050. On the other hand, with a tax rate fixed at about 8%, the replacement rate decreases steadily going from 45% in 1950 to about 14% in 2050.

(c) *Endogenous replacement rate*. With a PAYGO pension scheme that keeps the tax rate fixed in each period, the representative agent exploits the possibility to work more in periods when the demographic transition is particularly sharp (when, between 1990 and 2030, the growth rate of L_t/N_t is on a decreasing path) to smooth more per capita consumption. In this way the growth rate of the labor-population ratio decreases by less between 1990 and 2030 (cf. dashed line in Figure 14) which translates into a path of the real interest rate which is always higher than in the baseline and the case of exogenous labor with retirement age at 80 (cf. dashed line in Figure 13). Between 1990 and 2030 the real interest rate decreases only about 0.45 percentage points (a gain of about 0.30 percentage points compared to the case of exogenous labor with retirement age at 80).

(d) *Constant replacement rate*. The increase in the tax rate implied by a policy that keeps the replacement rate fixed in each period disincentivizes the supply of labor compared to the case (c) of fixed tax rate. Therefore, the path of the labor-population ratio growth rate is always at a lower level between 1990 and 2030 (compare the dashed and the dotted line in Figure 14). Moreover, the disincentive linked to the increasing tax rate is such that the labor-population ratio growth rate ends up being very close to the case of exogenous labor with retirement age at 80 – except for fluctuations associated to variations of the wedge attached to the aggregate disutility of labor ($\widetilde{\nu}_t$). The resulting path of the real interest rate is very similar to the case of exogenous labor with retirement age at 80 (cf. dotted line in Figure 13), slightly higher between 2000 and 2030 mostly due to endogenous effects that decrease the saving rate.[24]

4.3.4 Age-dependent labor productivity

Figure 7a has already shown that the age-dependent productivity, captured by the parameter h_j (cf. Figure 3), plays a quantitative important role. Here it is analyzed further. Figure 16 compares the path of the real interest rate in the baseline with the hypothetical case that all people in the labor force are equally productive, i.e. instead of using the baseline $L_t = \sum_{j=0}^{jr} h_j N_{t,j}$ it is assumed $L_t = \sum_{j=0}^{jr} N_{t,j}$. The results show that

[24] Notice that the result that the real interest rate ends up being higher in the case of endogenous replacement rate compared to the case of fixed replacement rate contrasts with the literature built on OLG models (cf. e.g. Krueger and Ludwig (2007)). The reason is that in the context of the representative household's model used here, the aggregate transfers associated with the pension scheme do not enter directly in the optimal condition for the saving decision. The impact of transfers to the representative household on choice variables is only indirect, via the implied path for the income tax rate and how this impacts the choice of labor supply.

the fact that the age distribution was in favor of relatively younger more productive workers has played a significant role in pushing up the real interest rate (compare the continuous line with the dash-dotted line). However, going further after about year 2017, the model predicts that there is a further drop in the real interest rate due to the fact that people in the labor force become older than before, i.e. relatively less productive.

The remaining two series show the result of another experiment. Suppose that the retirement age is increased by 10 years, going from age 65 to 75. Under the hump-shaped productivity profile of Figure 3, this measure would dampen the negative effect of demographic change on the real interest rate only by a small degree: the cohorts of people that now supply labor additionally are older, less productive than the average, so that their impact in increasing the effective labor supply is small. When it is assumed that people are equally productive, with retirement age at 75, the dampening effect of the demographic change on the real interest rate is reduced a lot. Comparing the dashed line with the dotted line in Figure 16, one can see the potency of a policy that could increase the productivity of older cohorts. Between 1990 and 2030 the real interest rate decreases by only about 0.20 percentage points in the scenario of equal productivity with retirement age at 75. This scenario can be taken as an extreme upper bound in the estimates provided, given that both the assumption of equal productivity across ages and a retirement age at 75 go against the data.

5 Conclusion

By means of an aggregate model, that approximates the solution path of an overlapping generation (OLG) model, this paper finds that demographic change has a significant impact on the natural real interest rate for the euro area: an upward pressure in the 70s and 80s and a prolonged downward pressure that extends at least until 2030 as the aging process unfolds (according to UN (2017) demographic projections). The model predicts in the baseline a decrease of the natural real interest rate of about 1 percentage point from 1990 to 2030 (roughly the peak to trough in the simulation) and the range of estimates lies in-between -1.7 and -0.4 percentage points according to a set of sensitivity specifications. The estimates suggest that the downward impact of aging could be mitigated not only by higher substitutability in production between labor and capital and higher intertemporal elasticity of substitution in consumption, but also by reforms aiming particularly at increasing the relative productivity of older cohorts and the participation rate. Increasing the retirement age per se has only a limited mitigating effect. There are two drivers explaining why aging has a downward impact on the natural real interest rate: labor as a production input becomes scarcer and individuals increase their willingness to save in anticipation of longer life expectancy. Both drivers are found to account about equally in explaining the downward trend of the natural real interest rate over the projected horizon. The fact that the saving rate decreases as the fraction of people retiring increases has a mitigating effect but never strong enough to counterbalance the two dampening drivers. Key to understand the impact of aging on the natural real interest rate is the evolution of the growth rate of the effective labor-population ratio, that is the ratio of the number of people in the working-age evaluated according to the age-dependent productivity over the number of people in the entire population.

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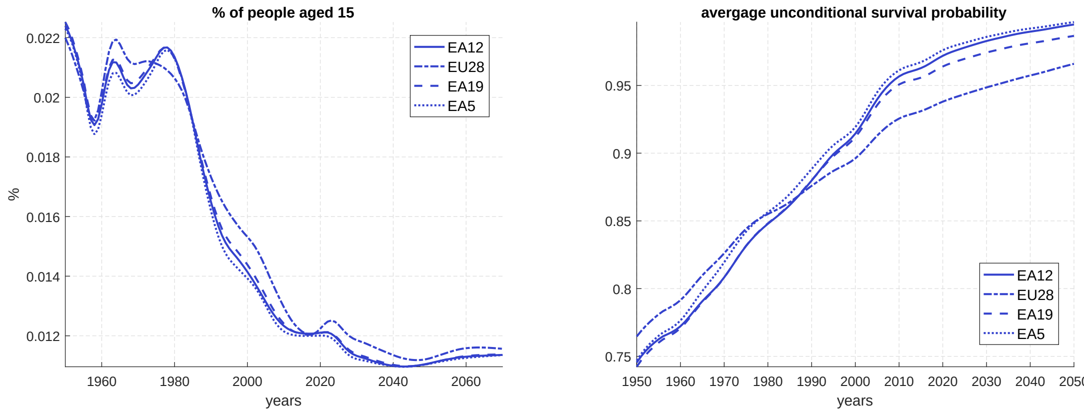
6 Figures



(a) Fraction of people aged 15 (b) Average unconditional survival probability

Figure 1: Demographic transition in Europe

Note. The indicator in panel 1a is the number of people aged 15 over the number of people aged more than 15. The indicator in panel 1b is computed by first retrieving the implied unconditional survival probabilities $\pi_{t,j}$ applying the recursive formula $N_{t+j,j} = \pi_{t+j,j} N_{t,0}$ using data for the cohort size $N_{t,j}$ for each year t and age-bin j (with $N_{t,0}$ corresponding to the incoming cohort size, those aged 15); then, by averaging across cohorts for each year so that the indicator is $\zeta_t = \sum_j \pi_{t,j}(N_{t,j}/N_t)$ with population size $N_t = \sum_j N_{t,j}$. Data from the United Nations (UN, 2017) *World Population Prospects: The 2017 Revision*, medium variant after year 2016 (cf. footnote 15). The following groups of countries hold. EA19: Austria, Belgium, Cyprus, Estonia, Finland, France, Germany, Greece, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, Netherland, Portugal, Slovakia, Slovenia, Spain; EA12 is EA19 excluding Cyprus, Estonia, Latvia, Lithuania, Malta, Slovakia, Slovenia; EA5: France, Germany, Italy, Netherlands, Spain; EU28 comprises EA19 and the following non-EA members: Bulgaria, Croatia, Czech Republic, Hungary, Poland, Romania, Sweden, Denmark and United Kingdom.

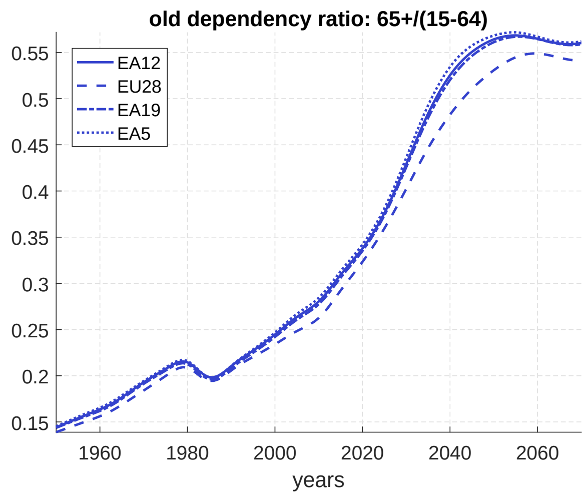


Figure 2: Old dependency ratio in Europe

Note. The indicator in the figure is the the number of people aged more than 64 over the number of people aged between 15 and 64. Data source and groups as in Figure 1a.

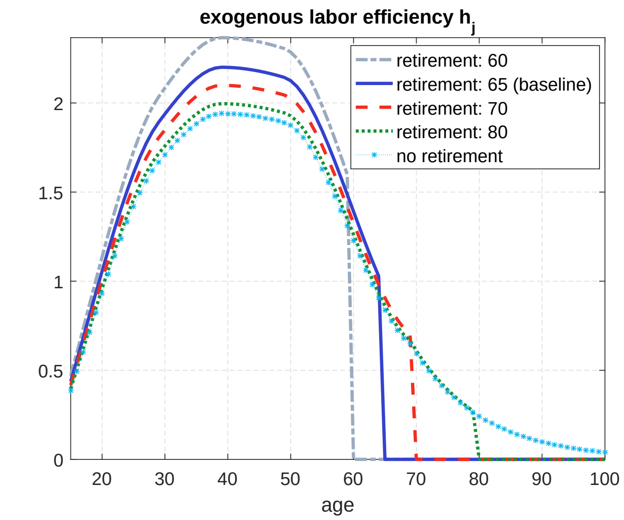


Figure 3: Age dependent labor supply in efficiency units, h_j

Note. The profile is obtained with a cubic interpolation (for age 15 to 70) on the data points provided in Domeij and Floden (2006). These data points are the product of participation rates provided by Fullerton (1999) and productivity provided by Hansen (1993). Lacking data, for $j \geq 70$ the profile is obtained from the following logistic function: $C/(1+Ae^{-Bj})$, with $A = .49$, $C = 50$, $B = (1/70)\log[h_{70}A/(C-h_{70})]$. The blue continuous line denotes the baseline profile with exogenous retirement age at $jr+1 = 65$. For each exogenous retirement age, the productivity profile h_j is normalized across the ages such that its mean is equal to 1.

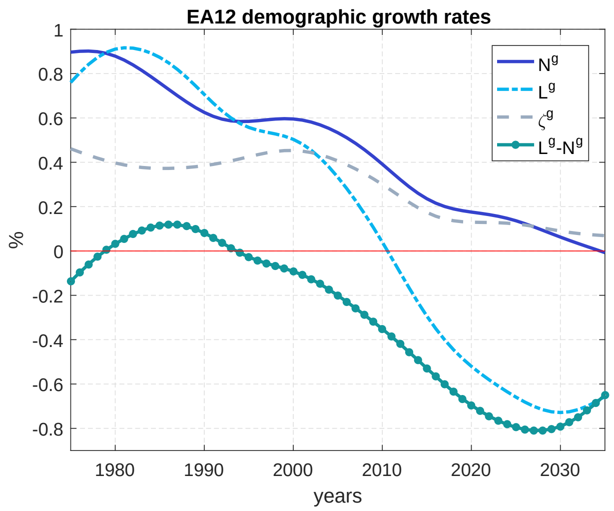


Figure 4: Demographic growth rates in EA12

Note. The number of people $N_{t,j}$ is taken from data provided by the United Nations (UN, 2017) *World Population Prospects: The 2017 Revision* for year $t \in [1950, 2100]$, medium variant after year 2016 (cf. footnote 15). The series plotted are N_t^g, L_t^g, ζ_t^g in (4.1) and (4.2), in net terms multiplied by 100, denoting the growth rates of population (people aged between 15 and 100), labor in efficiency units and aggregate unconditional survival probability, respectively.

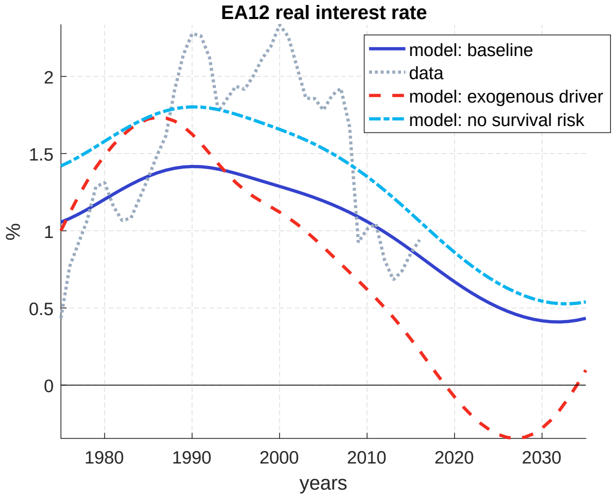


Figure 5: Real interest rate in EA12: $r_t \times 100$ [Section 4.2: Main Results]

Note. The series "model: baseline" is the result of the perfect-foresight simulation of the model composed by the system of four equations, (3.1)–(3.4), where the unique exogenous time-varying variables are N_t^g, L_t^g, ζ_t^g, cf. (4.1), (4.2). The series "data" refers to the net marginal product of capital computed applying a result from the model's optimal conditions: $r_t = (Y_t/K_t)\psi - \delta$, where values for ψ and δ are 0.3 and 0.095, respectively. Data for aggregate output Y_t and capital stock K_t are the same used for calibration, particularly they are "GDP (constant LCU)" for output and "Gross fixed capital formation (constant LCU)" for investment both sourced from WDI 2017, where capital is computed as explained in footnote 13. The series "model: exogenous driver" is the exogenous component in (3.4), i.e. $(L_t^g)^\sigma/(\beta\zeta_t^g(N_t^g)^\sigma) - 1$. The series "model: no survival risk" is obtained by setting $\zeta_t^g = 1$ for all t in the simulation.

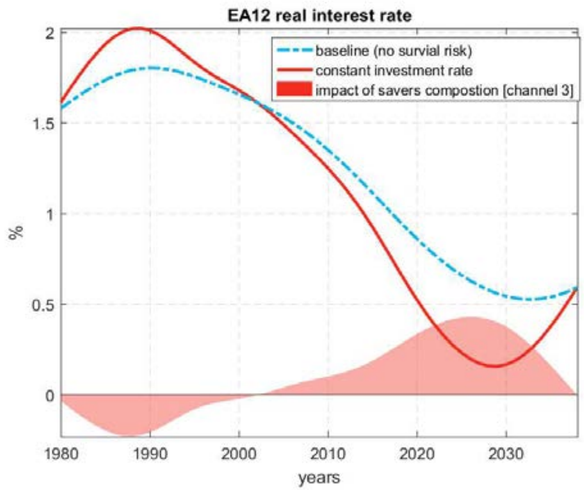
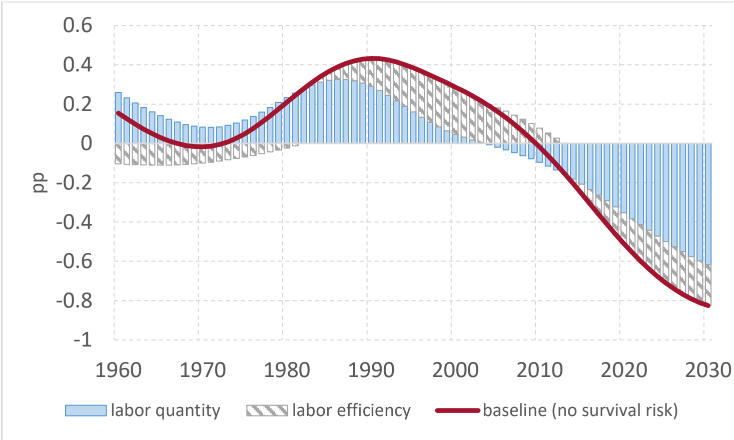


Figure 6: Real interest rate in EA12: impact of channel 3 [cf. Section 3.3]

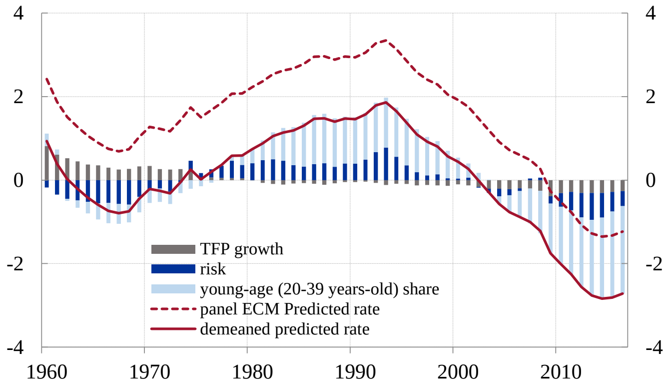
Note. The series "baseline (no survival risk)" is the dashed-dotted line in Figure 5. The series "constant investment rate" is obtained by running the simulation of the model assuming that the investment rate is fixed at the initial value of $\iota_t = \iota = 0.25$ for all periods t. Considering equations (3.9) and (3.10), the equilibrium real interest rate is in this case:

$$1 + r_{t+1} = \frac{(L_{t+1}^g)^\sigma}{\beta\zeta_{t+1}^g(N_{t+1}^g)^\sigma}\left(\frac{\widetilde{K}_t}{\widetilde{K}_{t-1}}\right)^{\psi\sigma}$$

where the capital-labor ratio $\widetilde{K}_t$ is such that $(\widetilde{K}_t L_{t+1}^g - (1-\delta)\widetilde{K}_{t-1})/\widetilde{K}_{t-1}^\psi = \iota$. The shaded area is the difference between the two curves



(a) Theoretical model ("model: no survival risk" in Figure 5)



(b) Econometric model (cf. WGEM (2018))

Figure 7: Natural real interest rate, demeaned: theoretical vs econometric model: drivers

***Note* to 7a**. The real interest rate path of the model is obtained with the "no survival risk" specification of Figure 5, i.e. by setting $\zeta_t^g = 1$ for all t in the simulation (this is done for the sake of comparison with the econometric model below, where demographics is captured merely by the the share of young-age people in the population). Therefore the only exogenous drivers are N_t^g and L_t^g. Since the growth rate of labor in efficiency units (L_t^g) depends on two parameters, their different impact is considered in isolation: "labor quantity" denotes the impact of the mere number of people in the labor force (aged between 15 and 64); "labor efficiency" denotes the impact of the age-varying productivity (technically, it is obtained by running the model twice: first, when L_t is the actual one: $\sum_{j=0}^{jr} h_j N_{t,j}$; second, considering $L_t = \sum_{j=0}^{jr} N_{t,j}$, i.e. by setting $h_j = 1$ for all j. Finally, the difference between the two implied curves is taken). All the implied series are demeaned over the period 1960–2030 in order to produce the standardized results in the figure.

***Note* to 7b**. Estimates of the natural real interest rate provided by Fiorentini, Galesi, Pérez-Quirós, and Sentana (2018) for the WGEM (2018)'s report from a panel error correction model (ECM) at annual frequency over the period 1899-2016. The unbalanced panel of advanced economies includes the following 17 countries: Australia, Belgium, Canada, Denmark, Finland, France, Germany, Italy, Japan, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom, and US. The observed real rate serves as dependent variable in the ECM, while some indicators about total factor productivity, demographics and risk serve as regressors. Data on TFP growth comes from Penn World Tables and Total Economy Database by The Conference Board https://www.conference-board.org/data/economydatabase/; data on demographic composition come from the Human-Mortality-Database http://mortality.org. The spread between long-term and short-term interest rate is used as proxy for the term premium to measure the time-varying risk aversion of agents. Interest rates data come from the Jordá, Knoll, Kuvshinov, Schularick, and Taylor (2017) Macro history Database and from the OECD Main Economic Indicators database.

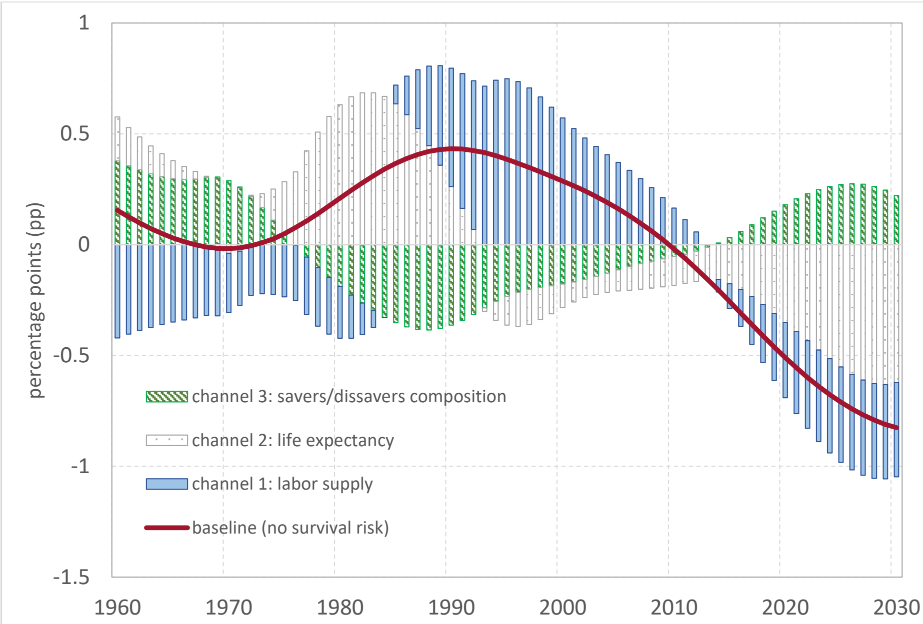


Figure 8: Real interest rate in EA12, demeaned: drivers [cf. Section 3.3]

Note. All the series implied by the simulation of the model are demeaned over the period 1960-2030 in order to have the standardized results in the figure. The series "baseline (no survival risk)" is the dashed-dotted line in Figure 5, i.e. the real interest rate under the specification with $\zeta_t^g = 1$ for all t. The series "channel 3: savers/dissavers composition" is the difference between the two series in Figure 6, i.e. the difference between the "baseline (no survival risk)" and the series implied by the model with constant investment ratio. The series "channel 2: life expectancy" is obtained by setting the growth rate of the number of people in the working age (age: 15–64) to 1 (which implies $L_t^g = 1$ for all t) using the implied growth rate of population in the simulation: in this way, all the change in the growth rate of the population comes from the fact that the number of elderly (aged more than 64) varies over time. The series "channel 1: labor supply" is the residual series, i.e. it is the difference between the "channel 3: savers/dissavers composition" and "channel 2: life expectancy": what is not explained by time-varying investment rate and number of elderly must be explained by the time-varying number of people in the working-age (and how this interacts with the age-varying productivity) as it is the only remaining driver.

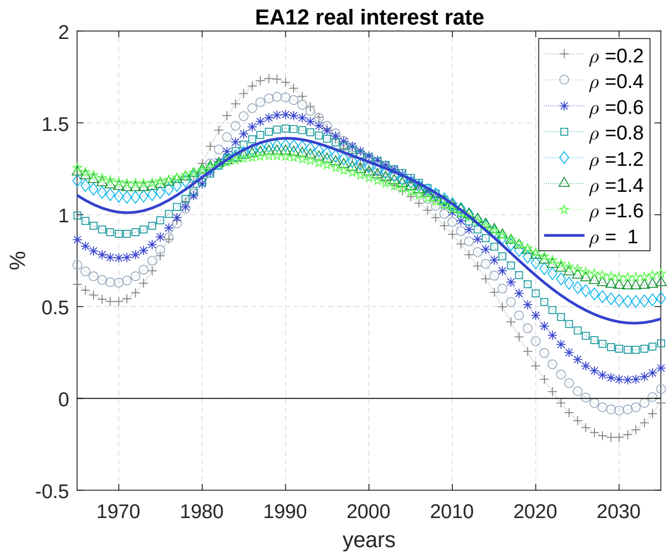


Figure 9: Real interest rate in EA12 (r_t): sensitivity to capital-labor elasticity (ρ)

Note. Perfect-foresight simulation of the model composed by the system of equations (4.4)–(4.8), for different values of ρ (elasticity of substitution between labor and capital) where the unique exogenous time-varying variables are N_t^g, L_t^g, ζ_t^g, cf. (4.1), (4.2).

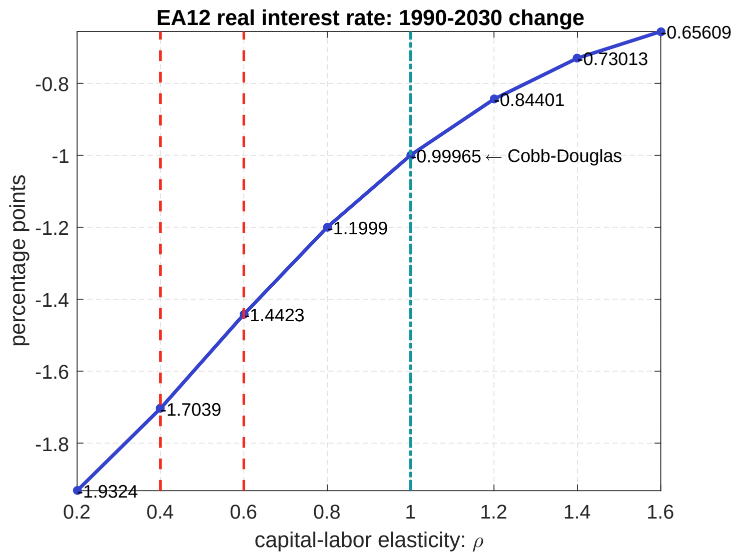


Figure 10: Real interest rate in EA12 (r_t): sensitivity to capital-labor elasticity (ρ): 1990–2030 change

Note. Percentage points change of the real interest rate between 1990 and 2030 as a function of the elasticity of substitution between labor and capital (ρ). Implied values from Figure 9.

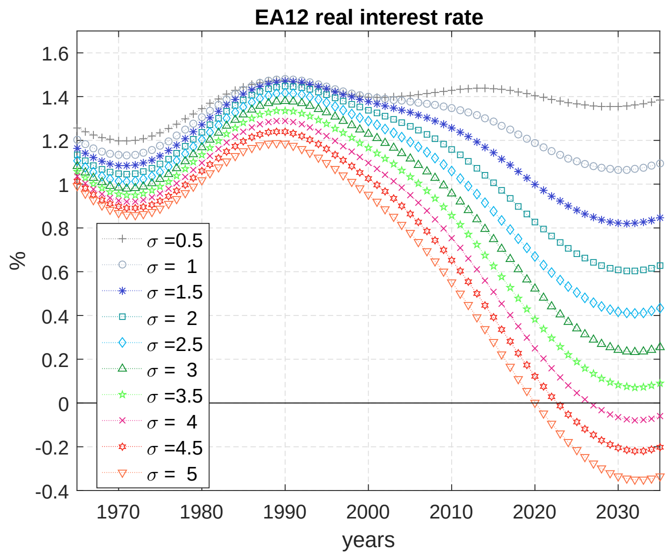


Figure 11: Real interest rate in EA12 (r_t): sensitivity to risk aversion (σ): 1990–2030 change

Note. Perfect-foresight simulation of the model composed by the system of equations (4.4)–(4.8), for different values of σ (constant relative risk aversion, CRRA) where the unique exogenous time-varying variables are N_t^g, L_t^g, ζ_t^g, cf. (4.1), (4.2).

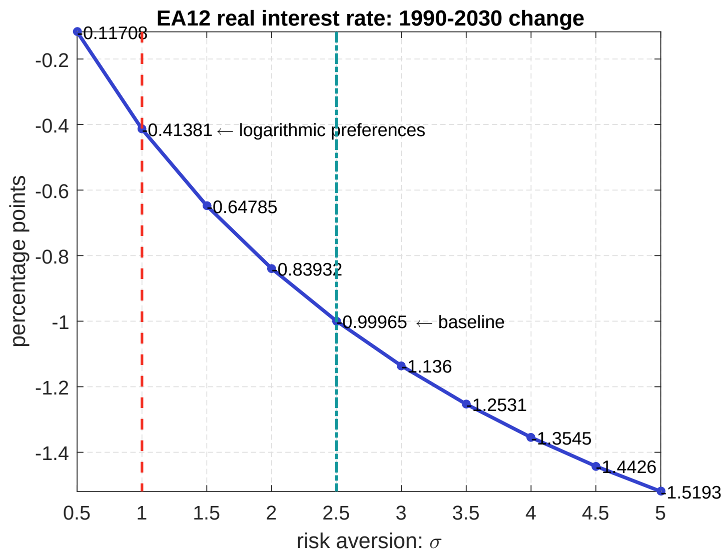


Figure 12: Real interest rate in EA12 (r_t): sensitivity to risk aversion (σ): 1990–2030 change

Note. Percentage points change of the real interest rate between 1990 and 2030 as a function of the CRRA coefficient (σ). Implied values from Figure 11.

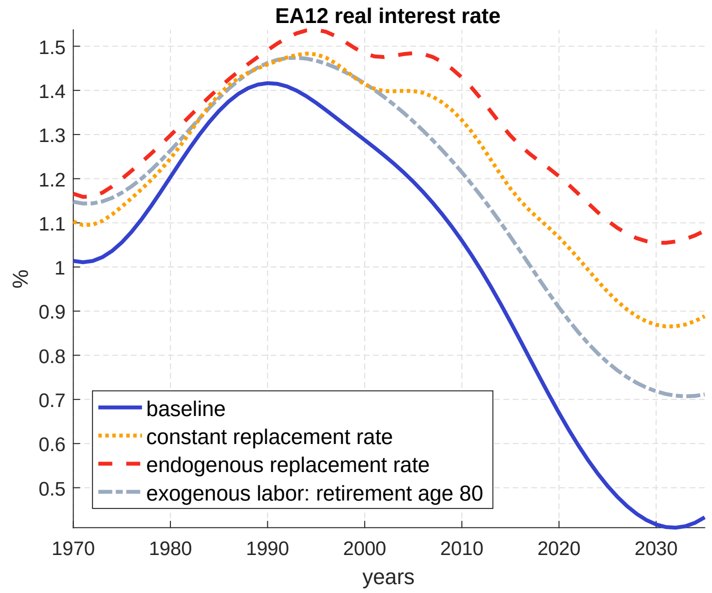


Figure 13: Real interest rate in EA12: baseline vs endogenous labor supply with PAYGO pension system and higher retirement age

Note. Simulation of the model described in section 4.3.3. Under the scenario *constant replacement rate* the tax rate τ_t varies endogenously to keep the government budget balanced in each period with replacement rate $\overline{d} = 0.45$; under the scenario *endogenous replacement rate* the tax rate τ_t is fixed at the initial steady state value $\tau_t = 0.0815$ (corresponding to a replacement rate of 45%) while the replacement rate varies endogenously to keep the government budget balanced in each period. The continuous line represents the same baseline simulation of Figure 5. The dash-dotted line is the result of assuming the retirement age at 80 instead of 65 (baseline), i.e. shifting jr by 15 age-bins forward in the exogenous labor supply $L_t = \sum_{j=0}^{jr} h_j N_{t,j}$.

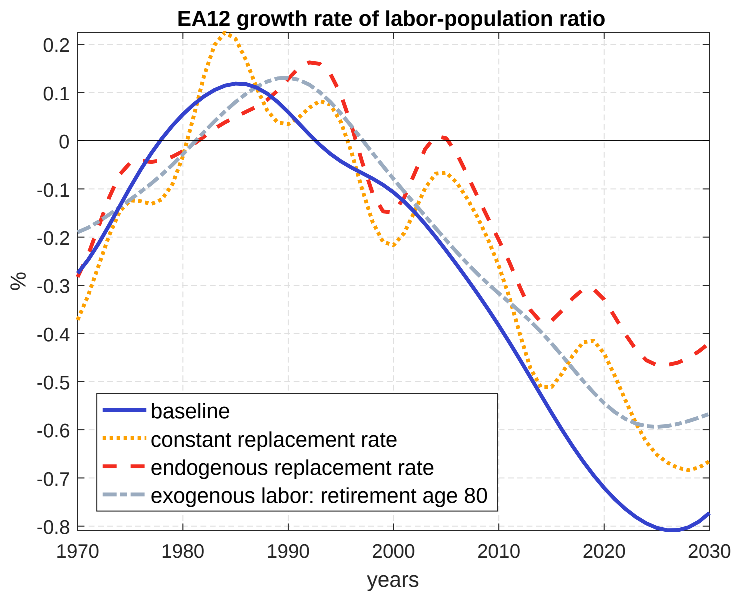


Figure 14: EA12 growth rate of labor-population ratio: $L_t\widetilde{l}_t/N_t$

Note. Growth rate of labor-population ratio under the different scenarios of Figure 13.

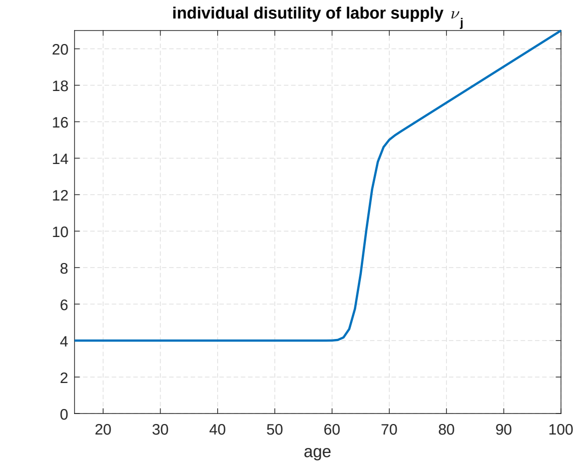


Figure 15: Age dependent disutility of labor supply, ν_j

Note. Following Jones (2018) (who uses a time-invariant specification of Mariano, Christopher, and Kathryn (2010)) the disutility of labor supply is given by the following expression:

$$\nu_j = b_0 + \left(b_1 \frac{j}{J+1}\right) \int_{-\infty}^{j} \frac{1}{(J+1)b_3\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{j-(J+1)b_2}{(J+1)b_3}\right)^2\right\} \mathrm{d}j$$

The parameter values chosen are: $b_0 = 4$, $b_1 = 17$, $b_2 = 0.6$, $b_3 = .02$. Notice that b_0 ensures that there is a baseline level of disutility which characterizes the function till age 60 approximately. $J + 1 = 86$ is the number of age-periods the individual can be alive (the individual enters the world at age 15 and remains alive up to age 100 at maximum). The integral expression is the normal cumulative distribution function over age j with mean $b_2(J + 1)$ and standard deviation $b_3(J + 1)$.

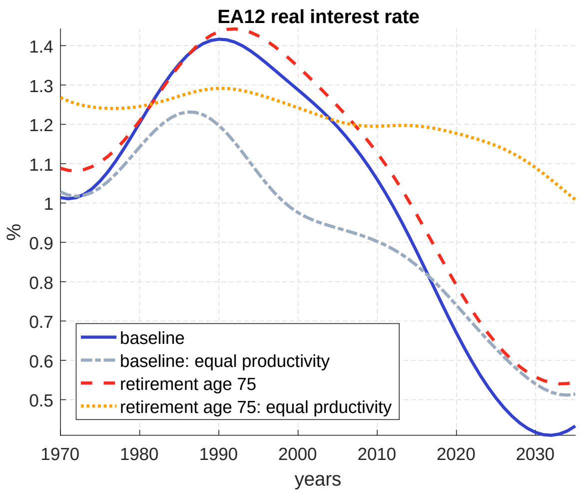


Figure 16: Real interest rate in EA12: role of age-dependent productivity h_j

Note. The series "baseline" is the result of the main model's simulation, the same continuous line of Figure 5. The series "baseline: equal productivity" is obtained by using $L_t = \sum_{j=0}^{jr} N_{t,j}$, i.e. by setting the age-dependent productivity parameter h_j (cf. Figure 3) equal to 1 for each age-bin $j = 0, 1, \cdots, jr$. The series "retirement age 75" is obtained by using $L_t = \sum_{j=0}^{jr} h_j N_{t,j}$ where jr has been increased from age 64 to age 74 (recall that h_j needs to be rescaled in order to take into account that the average productivity changes as the retirement age changes, cf. Figure 3). The series "retirement age 75: equal productivity" has jr corresponding to age 74, but now $L_t = \sum_{j=0}^{jr} N_{t,j}$, i.e. all people supplying labor exogenously are equally productive.

Appendix

A Aggregate representation of the OLG model with demographic wedges

This part of the appendix owes to Jones (2018)'s online appendix. Following Jones (2018), the derivation of the aggregate wedges that allow to represent the life-cycle (finite-life) individuals' problem in the OLG model (i.e. a problem where the heterogeneity across individuals is given by the age, cf. section 2.1) as a problem of an infinitely-lived representative agent problem proceeds in three steps. First, rewriting the individual's life-cycle problem as an infinite horizon problem, it is shown that under complete markets (namely, under the assumption of "perfect annuity market" for unintentional bequest, cf. footnote 4) there exist welfare weights attached to each individual utility function in the social planner's problem that allow to equate the planner's solution to the decentralized equilibrium. Second, by solving the social planner's dynamic problem of optimizing aggregate sectoral consumption, labor and savings over time, the decentralized equilibrium is related to the planner's solution. Third, by solving the social planner's static problem of choosing sectoral consumption and labor for each individual in each cohort to maximize the sum of individuals utilities weighted by the welfare weights, expressions for the aggregate demographic wedges attached to aggregate consumption and labor are derived.

1. Consider an individual i belonging to the cohort born in period j in a model where all agents are alive at time $t = 0$ and can trade claims to future consumption.[25] Rewrite his life-cycle problem as an infinite-horizon problem, to solve for each time-period t from the period he is born in s onward (till infinity), in the following way (consumption done over two sectors $\{T, N\}$):

$$\max_{\{c_t^{T,i,j}, c_t^{N,i,j}, l_t^{i,j}, a_{t+1}^{i,j}\}} \sum_{t=j}^{\infty} \beta^t \pi_{t,j} \phi_t^{i,j} u(c_t^{T,i,j}, c_t^{N,i,j}, l_t^{i,j})$$

$$\text{s.t.} \qquad a_{t+1}^{i,j} = \frac{a_t^{i,j}(1+r_t)}{s_t^{i,j}} - c_t^{T,i,j} - Z_t c_t^{N,i,j} + y_t^{i,j}$$

$$y_t^{i,j} = (1-\tau_t) w_t h^j l_t^{i,j} I(j \le jr) + d_{t,j} I(j > jr)$$

where $\pi_{t,j}$ denotes the unconditional survival probability in period t for an individual born in period j; as in Jones (2018), $\phi_t^{i,j}$ is a "preference process that proxies for the life-cycle" which takes value "one when the individual is alive and zero otherwise".[26] The remaining notation is the same as in section 2.1 except that now individuals are allowed to choose labor supply $0 < l_t^{i,j} < 1$ endogenously. Write the Lagrangian:

$$\mathcal{L} = \sum_{t=j}^{\infty} \beta^t \pi_{t,j} \left\{ \phi_t^{i,j} u(c_t^{T,i,j}, c_t^{N,i,j}, l_t^{i,j}) - \lambda_t^{i,j} \left[\frac{a_t^{i,j}(1+r_t)}{s_t^j} - c_t^{T,i,j} - Z_t c_t^{N,i,j} + y_t^{i,j} - a_{t+1}^{i,j} \right] \right\}$$

where the individual's Lagrangian multiplier $\lambda_t^{i,j}$ (the marginal utility of wealth) has been conveniently multiplied by $\pi_{t,j}$. The first order conditions are:

$$\begin{aligned}
\phi_t^{i,j} u_1(c_t^{T,i,j}, c_t^{N,i,j}, l_t^{i,j}) &= \lambda_t^{i,j} \\
\phi_t^{i,j} u_2(c_t^{T,i,j}, c_t^{N,i,j}, l_t^{i,j}) &= Z_t \lambda_t^{i,j} \\
\phi_t^{i,j} u_3(c_t^{T,i,j}, c_t^{N,i,j}, l_t^{i,j}) &= (1-\tau_t) w_t h^j \lambda_t^{i,j} \\
\lambda_t^{i,j} &= \beta(1+r_{t+1}) \lambda_{t+1}^{i,j}
\end{aligned}$$

[25] This makes the setting different from the one in section 2.

[26] In other terms, the problem is still a finite life one, because for each t greater than the terminal life period (J in the notation of section 2) $\phi_t^{i,j}$ is equal to zero in the problem above.

where the assumption of perfect annuity market gives the last equation, i.e. the standard Euler equation (independent of survival probabilities), given that $\frac{\pi_{t+1,j}}{\pi_{t,j}} = s^j_{t+1}$. Consider a different individual i' born in a different period j'. It follows for all i, i', $i \neq i'$:

$$\frac{\lambda^{i,j}_t}{\lambda^{i',j'}_t} = \frac{\lambda^{i,j}_{t+1}}{\lambda^{i',j'}_{t+1}} = \frac{\lambda^{i,j}_{t+2}}{\lambda^{i',j'}_{t+2}} = \cdots = \frac{\lambda^{i,j}}{\lambda^{i',j'}} \quad \text{for all } t$$

that is, the ratio of the marginal utilities of wealth of any two consumers is constant over time (which is a standard result under complete markets). This allows to represent the individual Lagrangian multipliers in the form $\lambda^{i,j}_t = \frac{\lambda_t}{\lambda^{i,j}}$, where λ_t is the Lagrangian multiplier on the aggregate budget constraint (which is identified later) and thus to map the social planner's solution to the decentralized equilibrium. Hence:

$$\begin{aligned}
\lambda^{i,j}\phi^{i,j}_t u_1(c^{T,i,j}_t, c^{N,i,j}_t, l^{i,j}_t) &= \lambda_t \\
\lambda^{i,j}\phi^{i,j}_t u_2(c^{T,i,j}_t, c^{N,i,j}_t, l^{i,j}_t) &= Z_t\lambda_t \\
\lambda^{i,j}\phi^{i,j}_t u_3(c^{T,i,j}_t, c^{N,i,j}_t, l^{i,j}_t) &= (1-\tau_t)w_t h^j \lambda_t
\end{aligned}$$

These three equations together with the each individual's budget constraint and aggregate definitions characterize the decentralized equilibrium.

2. Consider the social planner's dynamic problem of optimizing aggregate sectoral consumption, labor and savings over time:

$$\begin{aligned}
&\max_{C^T_t, C^N_t, l_t, K_t} \sum_{t=0}^{\infty} \beta^t U(C^T_t, C^N_t, l_t) \\
\text{s.t} \quad & C^T_t + Z_t C^N_t = (1-\tau_t)w_t l_t + (1+r_t)K_{t-1} - K_t + T_t
\end{aligned}$$

where T_t denotes any sort of transfer. Letting λ_t denoting the Lagrangian multiplier on the aggregate budget constraint, the first order conditions of this problem are the aggregate equivalent of those in the decentralised equilibrium:

$$\begin{aligned}
U_1(C^T_t, C^N_t, l_t) &= \lambda_t \\
U_2(C^T_t, C^N_t, l_t) &= Z_t\lambda_t \\
U_3(C^T_t, C^N_t, l_t) &= (1-\tau_t)w_t\lambda_t \\
\lambda_t &= \beta(1+r_t)\lambda_{t+1}
\end{aligned}$$

3. Consider the social planner's static problem of choosing sectoral consumption and labor for each individual in each cohort to maximize the sum of individuals utilities weighted by the welfare weights:

$$\begin{aligned}
U(C^T_t, C^N_t, l_t) &= \max_{c^T_{t,j}, c^N_{t,j}, l_{t,j}} \left\{ \sum_j \int \lambda^{i,j}\phi^{i,j}_t u(c^{T,i,j}_t, c^{N,i,j}_t, l^{i,j}_t) di \right\} \\
\text{s.t.} \quad C^T_t + Z_t C^N_t &= \sum_j \int c^{T,i,j}_t di + Z_t \sum_j \int c^{N,i,j}_t di \\
l_t &= \sum_j \int h_j l^{i,j}_t di
\end{aligned}$$

Recall that in the OLG model individuals within each cohort are identical. Moreover, within each cohort it is assumed that the mass of identical individuals is $N_{t,j}$ which denotes the (exogenous) number of people of age j at time t. It follows that each individual chooses $c_t^{T,i,j} \equiv c_{t,j}^T$, $c_t^{N,i,j} \equiv c_{t,j}^N$, $l_t^{i,j} \equiv l_{t,j}$ for all i. Hence, the social planner's problem becomes:

$$\begin{aligned} U(C_t^T, C_t^N, l_t) &= \max_{c_{t,j}^T, c_{t,j}^N, l_{t,j}} \left\{ \sum_j N_{t,j} \lambda^{i,j} \phi_t^{i,j} u(c_{t,j}^T, c_{t,j}^N, l_{t,j}) \right\} \\ \text{s.t.} \quad C_t^T + Z_t C_t^N &= \sum_j N_{t,j} c_{t,j}^T + Z_t \sum_j N_{t,j} c_{t,j}^N \\ l_t &= \sum_j N_{t,j} h_j l_{t,j} \end{aligned}$$

The following functional form is assumed:

$$u(c_{t,j}^T, c_{t,j}^N, l_{j,t}) = \frac{\left((c_{t,j}^T)^\gamma (c_{t,s}^N)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} - \nu_j \frac{(l_{j,t})^{1+\phi}}{1+\phi} \tag{A.1}$$

the Lagrangian for this static problem (with Lagrangian multipliers μ_t, ν_t) is:

$$\begin{aligned} \mathcal{L} &= \sum_s N_{t,s} \lambda^{i,s} \phi_t^{i,s} \left[\frac{\left((c_{t,s}^T)^{\alpha_s} (c_{t,s}^N)^{1-\alpha_s}\right)^{1-\sigma}}{1-\sigma} - \nu_j \frac{(l_{j,t})^{1+\phi}}{1+\phi} \right] + \mu_t \left[C_t^T + Z_t C_t^N - \sum_s N_{t,s} c_{t,s}^T - Z_t \sum_s c_{t,s}^N \right] \\ &+ \nu_t \left[l_t - \sum_j N_{t,j} h_j l_{t,j} \right] \end{aligned}$$

The optimal choice of $c_{t,j}^T$, $c_{t,s}^N$, $l_{t,j}$ leads to the first order conditions:

$$\begin{aligned} \lambda^{i,j} \phi_t^{i,j} \left((c_{t,j}^T)^\gamma (c_{t,j}^N)^{1-\gamma}\right)^{1-\sigma} \frac{\gamma}{c_{t,j}^T} &= \mu_t \\ \lambda^{i,j} \phi_t^{i,j} \left((c_{t,j}^T)^\gamma (c_{t,j}^N)^{1-\gamma}\right)^{1-\sigma} \frac{(1-\gamma)}{c_{t,j}^N} &= \mu_t Z_t \\ \lambda^{i,j} \phi_t^{i,j} \nu_j (l_{j,t})^{\phi-1} &= \nu_t h_j \end{aligned}$$

Combining the first two expressions above, the system becomes:

$$c_{t,j}^T = \frac{\gamma}{1-\gamma} Z_t c_{t,j}^N \tag{A.2}$$

$$\mu_t = \gamma \lambda^{i,j} \phi_t^{i,j} (c_{t,j}^N)^{(1-\gamma)(1-\sigma)} (c_{t,j}^T)^{\gamma(1-\sigma)-1} \tag{A.3}$$

$$\lambda^{i,j} \phi_t^{i,j} \nu_j (l_{j,t})^{\phi-1} = \nu_t h_j \tag{A.4}$$

Consider sectoral consumption first. Plug (A.2) into (A.3) to have:

$$\mu_t = \gamma \lambda^{i,j} \phi_t^{i,j} (c_{t,j}^N)^{(1-\gamma)(1-\sigma)} \left(\frac{\gamma}{1-\gamma} Z_t c_{t,j}^N \right)^{\gamma(1-\sigma)-1} = (c_{t,j}^N)^{-\sigma} \gamma \lambda^{i,j} \phi_t^{i,j} \left(\frac{\gamma Z_t}{1-\gamma} \right)^{\gamma(1-\sigma)-1}$$

i.e.

$$c_{t,j}^N = \left[\frac{\mu_t}{\gamma} \left(\frac{1-\gamma}{\gamma Z_t} \right)^{\gamma(1-\sigma)-1} \right]^{-\frac{1}{\sigma}} (\lambda^{i,j} \phi_t^{i,j})^{\frac{1}{\sigma}} \tag{A.5}$$

Similarly, by plugging (A.2) solved for $c^N_{t,j}$ into (A.3) the following expression holds true:

$$\mu_t = \gamma\lambda^{i,j}\phi^{i,j}_t \left(\frac{1-\gamma}{\gamma Z_t} c^T_{t,j}\right)^{(1-\gamma)(1-\sigma)} (c^T_{t,j})^{\gamma(1-\sigma)-1} = (c^T_{t,j})^{-\sigma}\gamma\lambda^{i,j}\phi^{i,j}_t \left(\frac{1-\gamma}{\gamma Z_t}\right)^{(1-\gamma)(1-\sigma)}$$

i.e.

$$c^N_{t,j} = \left[\frac{\mu_t}{\gamma}\left(\frac{1-\gamma}{\gamma Z_t}\right)^{(1-\gamma)(1-\sigma)}\right]^{-\frac{1}{\sigma}} (\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} \tag{A.6}$$

The goal is to find a utility function for the representative agent to let him choose only aggregate sectoral consumptions and labor. Thus, the goal is to find those (time-varying exogenous) parameters attached to aggregate sectoral consumption and labor that capture the change of the age structure in the economy. To this end, consider the aggregates using expressions (A.4) (A.5), (A.6):

$$\begin{aligned}
C^N_t &= \sum_j N_{t,j}c^N_{t,j} = \left[\frac{\mu_t}{\gamma}\left(\frac{1-\gamma}{\gamma Z_t}\right)^{\gamma(1-\sigma)-1}\right]^{-\frac{1}{\sigma}} \sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} = c^N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} \sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} \\
C^T_t &= \sum_j N_{t,j}c^T_{t,j} = \left[\frac{\mu_t}{\gamma}\left(\frac{1-\gamma}{\gamma Z_t}\right)^{(1-\gamma)(1-\sigma)}\right]^{-\frac{1}{\sigma}} \sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} = c^T_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} \sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}} \\
l_t &= \sum_j N_{t,j}h_j l_{t,j} = (\nu_t)^{\frac{1}{\phi}} \sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}} = (\lambda^{i,j}\phi^{i,j}_t\nu_j)^{\frac{1}{\phi}}(h_j)^{-\frac{1}{\phi}}(l_{j,t}) \sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}}
\end{aligned}$$

which imply that individual allocations are fractions of their respective aggregates:

$$c^T_{t,j} = \frac{(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}}{\sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}} C^T_t \tag{A.7}$$

$$c^N_{t,j} = \frac{(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}}{\sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}} C^N_t \tag{A.8}$$

$$l_{t,j} = \frac{(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}}(h_j)^{\frac{1}{\phi}}}{\sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}}} l_t \tag{A.9}$$

To find the aggregate representation, plug (A.7), (A.8), (A.9) into the objective function of the representative agent:

$$\begin{aligned}
\sum_j N_{t,j}\lambda^{i,j}\phi^{i,j}_t u(c^T_{t,j}, c^N_{t,j}, l_{t,j}) &= \sum_j N_{t,j}\lambda^{i,j}\phi^{i,j}_t \frac{\left((c^T_{t,j})^{\gamma}(c^N_{t,j})^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} - \sum_j N_{t,j}\lambda^{i,j}\phi^{i,j}_t\nu_j \frac{(l_{j,t})^{1+\phi}}{1+\phi} \\
&= \sum_j N_{t,j}\lambda^{i,j}\phi^{i,j}_t \frac{\left(\left(\frac{(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}}{\sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}} C^T_t\right)^{\gamma}\left(\frac{(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}}{\sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}} C^N_t\right)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} \\
&- \sum_j N_{t,j}\lambda^{i,j}\phi^{i,j}_t\nu_j \frac{\left(\frac{(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}}(h_j)^{\frac{1}{\phi}}}{\sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}}} l_t\right)^{1+\phi}}{1+\phi} \\
&= \frac{[(C^T_t)^{\gamma}(C^N_t)^{1-\gamma}]^{1-\sigma}}{1-\sigma}\left[\sum_j N_{t,j}(\lambda^{i,j}\phi^{i,j}_t)^{\frac{1}{\sigma}}\right]^{\sigma} \\
&- \frac{(l_t)^{1+\phi}}{1+\phi}\left[\sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi^{i,j}_t\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi}
\end{aligned}$$

Therefore, the two demographic exogenous wedges attached to aggregate consumption and labor are:

$$\varphi_t \equiv \left[\sum_j N_{t,j}(\lambda^{i,j}\phi_t^{i,j})^{\frac{1}{\sigma}}\right]^{\sigma} \tag{A.10}$$

$$\nu_t \equiv \left[\sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi_t^{i,j}\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi} \tag{A.11}$$

One can rewrite the choice of aggregate labor supply as a choice of the fraction of the exogenously available aggregate efficiency units of labor ($L_t = \sum_j h_j N_{t,j}$): $\widetilde{l}_t \equiv l_t/L_t$. That is, the part pertaining to labor in the objective function of the social planner can be equivalently rewritten in the following way:

$$\frac{(l_t)^{1+\phi}}{1+\phi}\left[\sum_j N_{t,j}(h_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi_t^{i,j}\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi} = \frac{(\widetilde{l}_t)^{1+\phi}}{1+\phi}\left[\sum_j N_{t,j}(\widetilde{h}_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi_t^{i,j}\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi}$$

where $\widetilde{h}_j = h_j/L_t$. Have:

$$\widetilde{\nu}_t \equiv \left[\sum_j N_{t,j}(\widetilde{h}_j)^{1+\frac{1}{\phi}}(\lambda^{i,j}\phi_t^{i,j}\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi} \tag{A.12}$$

In sum, the social planner's problem that allows to approximate the heterogeneity by age given by the OLG model (given the utility's functional form in (A.1)) is the following:

$$\max_{\{C_t^T, C_t^N, \widetilde{l}_t, K_t\}} \sum_{t=0}^{\infty}\beta^t\left\{\varphi_t\frac{[(C_t^T)^{\gamma}(C_t^N)^{1-\gamma}]^{1-\sigma}}{1-\sigma} - \widetilde{\nu}_t\frac{(\widetilde{l}_t)^{1+\phi}}{1+\phi}\right\} \tag{A.13}$$

$$\text{s.t.} \quad C_t^T + Z_t C_t^N + K_t = (1-\tau_t)w_t L_t \widetilde{l}_t + (1+r_t)K_{t-1} + T_t$$

where the exogenous wedges φ_t, $\widetilde{\nu}_t$ are given by (A.10) and (A.12), respectively. □

In the setting of section 2 the representative agent chooses also the aggregate sectoral labor supply for given aggregate efficiency units of labor (L_t). To prove that choosing the individual sectoral hours ($h_{t,s}^T$, $h_{t,s}^N$) is equivalent to choose the aggregate hours (L_t^T, L_t^N) under the CES aggregator for the social planner, consider the social planner's static problem of choosing sectoral aggregate labor for each individual in each cohort to maximize the sum of individuals utilities weighted by the welfare weights:

$$U(C_t^T, C_t^N, L_t^T, L_t^N) = \max_{c_{t,s}^T, c_{t,s}^N, h_{t,s}^T, h_{t,s}^N}\left\{\sum_s N_{t,s}\lambda^{i,s}\phi_t^{i,s}u(c_{t,s}^T, c_{t,s}^N)\right\}$$

$$\text{s.t.} \quad \sum_s N_{t,s}c_{t,s}^T + Z_t\sum_s N_{t,s}c_{t,s}^N = (1-\tau_t)[w_t^T L_t^T + w_t^N L_t^N] + \cdots$$

$$\sum_s \underbrace{\left[\chi^{-\frac{1}{\varepsilon}}(h_{t,s}^T)^{\frac{\varepsilon+1}{\varepsilon}} + (1-\chi)^{-\frac{1}{\varepsilon}}(h_{t,s}^N)^{\frac{\varepsilon+1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon+1}}}_{h_s} N_{t,s} = L_t$$

$$L_t^T = \sum_s h_{t,s}^T N_{t,s}$$

$$L_t^N = \sum_s h_{t,s}^N N_{t,s}$$

The first order conditions with respect to $h^T_{t,s}$, $h^N_{t,s}$ are (exactly equal to those in the decentralized equilibrium):

$$
\begin{aligned}
h^T_{t,s} &= \chi h_s \left(\frac{w^T_t}{w_t}\right)^{\varepsilon} \\
h^N_{t,s} &= (1-\chi) h_s \left(\frac{w^N_t}{w_t}\right)^{\varepsilon}
\end{aligned}
$$

where

$$
w_t = \left[\chi (w^T_t)^{1+\varepsilon} + (1-\chi)(w^N_t)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}}
$$

The goal now is to prove that the following holds:

$$
L_t = \sum_s \left[\chi^{-\frac{1}{\varepsilon}} (h^T_{t,s})^{\frac{\varepsilon+1}{\varepsilon}} + (1-\chi)^{-\frac{1}{\varepsilon}} (h^N_{t,s})^{\frac{\varepsilon+1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon+1}} N_{t,s} = \left[\chi^{-\frac{1}{\varepsilon}} (L^T_t)^{\frac{\varepsilon+1}{\varepsilon}} + (1-\chi)^{-\frac{1}{\varepsilon}} (L^N_t)^{\frac{\varepsilon+1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon+1}}
$$

Start from the definition of aggregate hours, plugging into the individual's first order conditions:

$$
\begin{aligned}
L^T_t &= \sum_s h^T_{t,s} N_{t,s} = \chi \left(\frac{w^T_t}{w_t}\right)^{\varepsilon} \sum_s h_s N_{t,s} \\
L^N_t &= \sum_s h^N_{t,s} N_{t,s} = (1-\chi) \left(\frac{w^N_t}{w_t}\right)^{\varepsilon} \sum_s h_s N_{t,s}
\end{aligned}
$$

that imply:

$$
\begin{aligned}
w^T_t &= \left(\frac{L^T_t}{\chi \sum_s h_s N_{t,s}}\right)^{\frac{1}{\varepsilon}} \\
w^N_t &= \left(\frac{L^N_t}{(1-\chi) \sum_s h_s N_{t,s}}\right)^{\frac{1}{\varepsilon}}
\end{aligned}
$$

plug the last two expressions into the expression for the wage (which is implied by the individual's problem), $w_t = \left[\chi (w^T_t)^{1+\varepsilon} + (1-\chi)(w^N_t)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}}$, to have:

$$
L_t = \sum_s h_s N_{t,s} = \left[\chi^{-\frac{1}{\varepsilon}} (L^T_t)^{\frac{\varepsilon+1}{\varepsilon}} + (1-\chi)^{-\frac{1}{\varepsilon}} (L^N_t)^{\frac{\varepsilon+1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon+1}}
$$

□

B Solving the model

B.1 Optimal conditions

The first order conditions of the representative household's problem (2.4) read:

$$
\begin{aligned}
\frac{\varphi_t \gamma (C_t^T)^{\gamma(1-\sigma)} (C_t^N)^{(1-\gamma)(1-\sigma)}}{C_t^T} &= \lambda_t \\
\frac{\varphi_t (1-\gamma) (C_t^T)^{\gamma(1-\sigma)} (C_t^N)^{(1-\gamma)(1-\sigma)}}{C_t^N} &= Z_t \lambda_t \\
(1+r_{t+1})\beta \zeta_{t+1} \lambda_{t+1} &= \zeta_t \lambda_t \\
L_t^T &= \chi L_t \left(\frac{w_t^T}{w_t} \right)^{\epsilon} \\
L_t^N &= (1-\chi) L_t \left(\frac{w_t^N}{w_t} \right)^{\epsilon} \\
w_t &= \left[\chi (w_t^T)^{\epsilon+1} + (1-\chi)(w^N)^{\epsilon+1} \right]^{\frac{1}{\epsilon+1}}
\end{aligned}
$$

while the first order conditions for the representative firms (2.11) and (2.12) are:

$$
\begin{aligned}
\psi (A_t^T)^{1-\psi} \left(\frac{K_t^T}{L_t^T} \right)^{\psi-1} &= r_t + \delta \\
(1-\psi)(A_t^T)^{1-\psi} \left(\frac{K_t^T}{L_t^T} \right)^{\psi} &= w_t^T \\
\psi Z_t (A_t^N)^{1-\psi} \left(\frac{K_t^N}{L_t^N} \right)^{\psi-1} &= r_t + \delta \\
(1-\psi) Z_t (A_t^N)^{1-\psi} \left(\frac{K_t^N}{L_t^N} \right)^{\psi} &= w_t^N
\end{aligned}
$$

where the no arbitrage condition (2.13) has been used. Together with clearing conditions (2.2), (2.14), (2.15) and (2.16) and production functions (2.9) and (2.10) there is a system of 16 equations with 16 unknowns: $\{\lambda_t, Z_t, r_t, w_t, w_t^N, w_t^T, C_t^T, C_t^N, L_t^T, L_t^N, L_t, K_t^T, K_t^N, K_t, Y_t^T, Y_t^N\}$.

It is convenient to divide quantities by the exogenous number of efficiency units of labor L_t. For each variable X_t have the following notation $\widetilde{X}_t \equiv X_t / L_t$. The exception is capital, because of its predetermined nature it will be $\widetilde{K}_{t-1} = K_{t-1}/L_t$. Then, the above first order conditions become:

$$
\begin{aligned}
\frac{\varphi_t \gamma (\widetilde{C}_t^T)^{\gamma(1-\sigma)} (\widetilde{C}_t^N)^{(1-\gamma)(1-\sigma)}}{L_t^{\sigma} \widetilde{C}_t^T} &= \lambda_t \\
\frac{\varphi_t (1-\gamma) (\widetilde{C}_t^T)^{\gamma(1-\sigma)} (\widetilde{C}_t^N)^{(1-\gamma)(1-\sigma)}}{L_t^{\sigma} \widetilde{C}_t^N} &= Z_t \lambda_t \\
(1+r_{t+1})\beta \zeta_{t+1} \lambda_{t+1} &= \zeta_t \lambda_t \\
\widetilde{L}_t^T &= \chi \left(\frac{w_t^T}{w_t} \right)^{\epsilon} \\
\widetilde{L}_t^N &= (1-\chi) \left(\frac{w_t^N}{w_t} \right)^{\epsilon} \\
w_t &= \left[\chi (w_t^T)^{\epsilon+1} + (1-\chi)(w^N)^{\epsilon+1} \right]^{\frac{1}{\epsilon+1}}
\end{aligned}
$$

$$
\begin{aligned}
\psi(A_t^T)^{1-\psi}\left(\frac{\widetilde{K}_t^T}{\widetilde{L}_t^T}\right)^{\psi-1} &= r_t+\delta \\
(1-\psi)(A_t^T)^{1-\psi}\left(\frac{\widetilde{K}_t^T}{\widetilde{L}_t^T}\right)^{\psi} &= w_t^T \\
\psi Z_t(A_t^N)^{1-\psi}\left(\frac{\widetilde{K}_t^N}{\widetilde{L}_t^N}\right)^{\psi-1} &= r_t+\delta \\
(1-\psi)Z_t(A_t^N)^{1-\psi}\left(\frac{\widetilde{K}_t^N}{\widetilde{L}_t^N}\right)^{\psi} &= w_t^N \\
\widetilde{Y}_t^T &= (\widetilde{K}_t^T)^{\psi}(A_t^T\widetilde{L}_t^T)^{1-\psi} \\
\widetilde{Y}_t^N &= (\widetilde{K}_t^N)^{\psi}(A_t^N\widetilde{L}_t^N)^{1-\psi} \\
\widetilde{K}_{t-1} &= \widetilde{K}_t^N+\widetilde{K}_t^T \\
\widetilde{C}_t^N &= \widetilde{Y}_t^N \\
\widetilde{C}_t^T+\widetilde{K}_tL_{t+1}^g &= (1-\delta)\widetilde{K}_{t-1}+\widetilde{Y}_t^T
\end{aligned}
$$

where $L_t^g \equiv L_{t+1}/L_t$ denotes the growth rate of the efficiency units of labor.
To have wage equalization in all periods:

$$w_t^T = w_t^N = w_t \tag{B.1}$$

for all t, it is assumed that there is perfect mobility of labor (i.e. $\epsilon \to \infty$). This implies $\widetilde{L}_t^N + \widetilde{L}_t^T = 1$. Hence, with the notation above, a time-varying fraction χ_t of the exogenous labor supply L_t is employed in the T-sector. Consequently, a fraction $(1-\chi_t)$ is employed in the N-sector. That is:

$$\widetilde{L}_t^T = \chi_t, \quad \widetilde{L}_t^N = 1-\chi_t \tag{B.2}$$

where χ_t is an endogenous variable. Then, by managing the above optimal conditions, the model can be characterized as follows:

$$
\begin{aligned}
\widetilde{C}_t^T &= \frac{\gamma}{1-\gamma}Z_t\widetilde{C}_t^N && \text{(B.3)}\\
Z_t &= \left(\frac{A_t^T}{A_t^N}\right)^{1-\psi} && \text{(B.4)}\\
w_t &= A_t^T(1-\psi)\left(\frac{\psi}{r_t+\delta}\right)^{\frac{\psi}{1-\psi}} && \text{(B.5)}\\
\widetilde{K}_{t-1} &= \frac{w_t\psi}{(r_t+\delta)(1-\psi)} && \text{(B.6)}\\
\widetilde{K}_t^T &= \widetilde{K}_{t-1}\chi_t && \text{(B.7)}\\
\widetilde{K}_t^N &= \widetilde{K}_{t-1}(1-\chi_t) && \text{(B.8)}\\
\widetilde{Y}_t^T &= \chi_t(A_t^T)^{1-\psi}\widetilde{K}_{t-1}^{\psi} && \text{(B.9)}\\
\widetilde{Y}_t^N &= (1-\chi_t)(A_t^N)^{1-\psi}\widetilde{K}_{t-1}^{\psi} && \text{(B.10)}\\
\widetilde{C}_t^N &= \widetilde{Y}_t^N && \text{(B.11)}\\
\widetilde{C}_t^T+\widetilde{K}_tL_{t+1}^g &= (1-\delta)\widetilde{K}_{t-1}+\widetilde{Y}_t^T && \text{(B.12)}
\end{aligned}
$$

The system can be further simplified. Consider the clearing condition in the market for T-goods (B.12), using (B.3), (B.4), (B.9), (B.10) and (B.11). It results:

$$\frac{\gamma}{1-\gamma}\left(\frac{A_t^T}{A_t^N}\right)^{1-\psi}(1-\chi_t)(A_t^N)^{1-\psi}\widetilde{K}_{t-1}^{\psi}+\widetilde{K}_tL_{t+1}^g \quad = \quad (1-\delta)\widetilde{K}_{t-1}+\chi_t(A_t^T)^{1-\psi}\widetilde{K}_{t-1}^{\psi}$$

i.e.

$$\frac{\gamma}{1-\gamma}(A_t^T)^{1-\psi}\widetilde{K}_{t-1}^{\psi}+\widetilde{K}_tL_{t+1}^g-(1-\delta)\widetilde{K}_{t-1} \quad = \quad \chi_t(A_t^T)^{1-\psi}\widetilde{K}_{t-1}^{\psi}\left(1+\frac{\gamma}{1-\gamma}\right)$$

i.e.

$$\chi_t \quad = \quad \gamma+(1-\gamma)\frac{[\widetilde{K}_tL_{t+1}^g-(1-\delta)\widetilde{K}_{t-1}]}{(A_t^T)^{1-\psi}\widetilde{K}_{t-1}^{\psi}} \tag{B.13}$$

Subsequently, plug (B.3) into the Euler equation:

$$(1+r_{t+1})\beta\zeta_{t+1}\frac{\varphi_{t+1}\gamma(\widetilde{C}_{t+1}^T)^{\gamma(1-\sigma)}(\widetilde{C}_{t+1}^N)^{(1-\gamma)(1-\sigma)}}{L_{t+1}^{\sigma}\widetilde{C}_{t+1}^T} \quad = \quad \zeta_t\frac{\varphi_t\gamma(\widetilde{C}_t^T)^{\gamma(1-\sigma)}(\widetilde{C}_t^N)^{(1-\gamma)(1-\sigma)}}{L_t^{\sigma}\widetilde{C}_t^T}$$

to have:

$$(1+r_{t+1})\beta\zeta_{t+1}\frac{\varphi_{t+1}\left(\frac{\gamma}{1-\gamma}Z_{t+1}\widetilde{C}_{t+1}^N\right)^{\gamma(1-\sigma)}(\widetilde{C}_{t+1}^N)^{(1-\gamma)(1-\sigma)}}{L_{t+1}^{\sigma}\frac{\gamma}{1-\gamma}Z_t\widetilde{C}_{t+1}^N} \quad = \quad \zeta_t\frac{\varphi_t\left(\frac{\gamma}{1-\gamma}Z_t\widetilde{C}_t^N\right)^{\gamma(1-\sigma)}(\widetilde{C}_t^N)^{(1-\gamma)(1-\sigma)}}{L_t^{\sigma}\frac{\gamma}{1-\gamma}Z_t\widetilde{C}_t^N}$$

i.e.

$$(1+r_{t+1})\beta\zeta_{t+1}\frac{\varphi_{t+1}\left(\frac{\gamma}{1-\gamma}Z_{t+1}\right)^{\gamma(1-\sigma)-1}(\widetilde{C}_{t+1}^N)^{-\sigma}}{L_{t+1}^{\sigma}} \quad = \quad \zeta_t\frac{\varphi_t\left(\frac{\gamma}{1-\gamma}Z_t\right)^{\gamma(1-\sigma)-1}(\widetilde{C}_t^N)^{-\sigma}}{L_t^{\sigma}}$$

i.e.

$$(1+r_{t+1})\beta\frac{\zeta_{t+1}^g\varphi_{t+1}^g}{(L_{t+1}^g)^{\sigma}}\left(\frac{Z_{t+1}}{Z_t}\right)^{\gamma(1-\sigma)-1} \quad = \quad \left(\frac{\widetilde{C}_{t+1}^N}{\widetilde{C}_t^N}\right)^{\sigma} \tag{B.14}$$

where $\varphi_{t+1}^g=\varphi_{t+1}/\varphi_t$, $L_{t+1}^g\equiv L_{t+1}/L_t$, $\zeta_{t+1}^g=\zeta_{t+1}/\zeta_t$. Given $Z_t=\left(A_t^T/A_t^N\right)^{1-\psi}$ and $\widetilde{C}_t^N=\widetilde{Y}_t^N=(1-\chi_t)(A_t^N)^{1-\psi}\widetilde{K}_{t-1}^{\psi}$ the last expression results:

$$(1+r_{t+1})\beta\frac{\zeta_{t+1}^g\varphi_{t+1}^g}{(L_{t+1}^g)^{\sigma}}\left(\frac{A_{t+1}^T/A_t^T}{A_{t+1}^N/A_t^N}\right)^{(1-\psi)(\gamma(1-\sigma)-1)} \quad = \quad \left(\frac{(A_{t+1}^N)^{1-\psi}(1-\chi_{t+1})\widetilde{K}_t^{\psi}}{(A_t^N)^{1-\psi}(1-\chi_t)\widetilde{K}_{t-1}^{\psi}}\right)^{\sigma}$$

i.e.

$$1+r_{t+1} \quad = \quad \frac{(L_{t+1}^g)^{\sigma}}{\beta\zeta_{t+1}^g(N_{t+1}^g)^{\sigma}}\left(\frac{\widetilde{K}_t}{\widetilde{K}_{t-1}}\right)^{\psi\sigma}\left(\frac{1-\chi_{t+1}}{1-\chi_t}\right)^{\sigma}\left(\frac{A_{t+1}^N}{A_t^N}\right)^{(1-\psi)\sigma}\left(\frac{A_{t+1}^N/A_t^N}{A_{t+1}^T/A_t^T}\right)^{(1-\psi)(\gamma(1-\sigma)-1)} \tag{B.15}$$

Recall that $\varphi_t=N_t^{\sigma}$, hence $\varphi_{t+1}^g=(N_{t+1}^g)^{\sigma}$ with $N_{t+1}^g\equiv N_{t+1}/N_t$. Equations (B.5), (B.6), (B.13) and (B.15) compose a system of four equations and four unknowns $(w_t, r_r, \chi_t, \widetilde{K}_t)$ fully characterizing the dynamic equilibrium of the model.

To isolate the impact of demographic change the analysis abstracts from technological change setting:

$$A_t^T = A^T, \quad A_t^N = A^N \quad \forall t$$

Hence, the expression for the real interest rate becomes:

$$1 + r_{t+1} = \frac{(L_{t+1}^g)^\sigma}{\beta \zeta_{t+1}^g (N_{t+1}^g)^\sigma} \left(\frac{\widetilde{K}_t}{\widetilde{K}_{t-1}} \right)^{\psi\sigma} \left(\frac{1 - \chi_{t+1}}{1 - \chi_t} \right)^\sigma$$

i.e.

$$1 + r_{t+1} = \frac{(L_{t+1}^g)^{\sigma(1-\psi)}}{\beta \zeta_{t+1}^g (N_{t+1}^g)^\sigma} \left(\frac{K_t}{K_{t-1}} \right)^{\psi\sigma} \left(\frac{1 - \chi_{t+1}}{1 - \chi_t} \right)^\sigma \tag{B.16}$$

recalling that $\widetilde{K}_t = K_t / L_{t+1}$.

Hence, with no technological change, the set of four equations that fully characterize the dynamic equilibrium (with endogenous variables $w_t, r_r, \chi_t, \widetilde{K}_t$) is:

$$w_t = A^T (1 - \psi) \left(\frac{\psi}{r_t + \delta} \right)^{\frac{\psi}{1-\psi}} \tag{B.17}$$

$$\widetilde{K}_{t-1} = \frac{w_t \psi}{(r_t + \delta)(1 - \psi)} \tag{B.18}$$

$$\chi_t = \gamma + (1 - \gamma) \frac{[\widetilde{K}_t L_{t+1}^g - (1 - \delta)\widetilde{K}_{t-1}]}{(A^T)^{1-\psi} \widetilde{K}_{t-1}^\psi} \tag{B.19}$$

$$1 + r_{t+1} = \frac{(L_{t+1}^g)^\sigma}{\beta \zeta_{t+1}^g (N_{t+1}^g)^\sigma} \left(\frac{\widetilde{K}_t}{\widetilde{K}_{t-1}} \right)^{\psi\sigma} \left(\frac{1 - \chi_{t+1}}{1 - \chi_t} \right)^\sigma \tag{B.20}$$

□

B.2 Steady states

Denote variables at the initial steady state with no time subscript. In the initial steady state it is assumed that $L_{t+1}^g = \varphi_{t+1}^g = \zeta_{t+1}^g = 1$ which implies that there is no dynamics in the model, i.e. each variable has always the same value, and from (B.20):

$$r = \frac{1}{\beta} - 1 \tag{B.21}$$

Then, from (B.17), (B.18) and (B.19):

$$w = A^T (1 - \psi) \left(\frac{\psi}{r + \delta} \right)^{\frac{\psi}{1-\psi}} \tag{B.22}$$

$$\widetilde{K} = \frac{w\psi}{(r + \delta)(1 - \psi)} \tag{B.23}$$

$$\chi = \gamma + (1 - \gamma) \frac{\delta \widetilde{K}^{1-\psi}}{(A^T)^{1-\psi}} \tag{B.24}$$

The value for all the remaining variable in the initial steady state is so identified (cf. (B.3)–(B.11)):

$$\widetilde{L}^T = \chi \tag{B.25}$$
$$\widetilde{L}^T = (1-\chi) \tag{B.26}$$
$$\widetilde{K}^T = \widetilde{K}\chi \tag{B.27}$$
$$\widetilde{K}^N = \widetilde{K}(1-\chi) \tag{B.28}$$
$$\widetilde{Y}^T = \chi(A^T)^{1-\psi}\widetilde{K}^{\psi} \tag{B.29}$$
$$\widetilde{Y}^N = (1-\chi)(A^N)^{1-\psi}\widetilde{K}^{\psi} \tag{B.30}$$
$$\widetilde{C}^N = \widetilde{Y}^N \tag{B.31}$$
$$Z = (A^T/A^N)^{1-\psi} \tag{B.32}$$
$$\widetilde{C}^T = \frac{\gamma}{1-\gamma} Z\widetilde{C}^N_t \tag{B.33}$$

If the exogenous demographic variables are allowed to have a different value from the one in the initial steady state (denote the variables with subscript F in the final steady state), i.e. if L^g_F, N^g_F, ζ^g_F can differ from 1, it follows that the value of the real interest rate can differ from the initial steady state:

$$r_F = \frac{(L^g_F)^{\sigma}}{\beta\zeta^g_F(N^g_F)^{\sigma}} - 1 \tag{B.34}$$

In this case, the final steady state value of the remaining variables follows directly by using equations (B.22)–(B.33) with r_F as new value for the real interest rate.

B.3 CES production function

Consider a constant elasticity of substitution (CES) production function in each sector $s \in \{T, N\}$:

$$Y^s_t = \left[\psi(K^s_t)^{\frac{\rho-1}{\rho}} + (1-\psi)(L^s_t)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}$$

where ψ is again the bias towards capital and ρ is the elasticity of substitution between labor (L^s_t) and capital (K^s_t). Recall that depending on the value of ρ there are two main cases:

(a) $\rho \leq 1$: capital and labor are *gross complements* in production, i.e. a lower supply of one input reduces the demand for the other. The special case is when $\rho \to 0$: the production function tends to a *Leontief*, i.e. output can only be produced using capital and labor in fixed proportions;

(b) $\rho > 1$: capital and labor are *gross substitutes*, i.e. a lower supply of one input creates added demand for the other. The special case is when $\rho \to \infty$: capital and labor are *perfect substitutes*.

When $\rho \to 1$ the production function is in the limit a Cobb-Douglas (used in the baseline specification above, cf. (2.9), (2.10)), with fixed shares paid to each factor.
Using again the notation $\widetilde{X}_t \equiv X_t/L_t$ for each variable X_t (for aggregate capital it will be $\widetilde{K}_{t-1} \equiv K_{t-1}/L_t$ because of its predetermined nature), the CES production function above can be easily written as:

$$\widetilde{Y}_t = \left[\psi(\widetilde{K}^s_t)^{\frac{\rho-1}{\rho}} + (1-\psi)(\widetilde{L}^s_t)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}$$

Under the assumption of wage equalization between sectors, $w_t^T = w_t^N = w_t$ for all t the first order conditions for the the firms' profit maximization problem read:

$$\left(\frac{\widetilde{Y}_t^T}{\widetilde{L}_t^T}\right)^{\frac{1}{\rho}}(1-\psi) = w_t \tag{B.35}$$

$$\left(\frac{\widetilde{Y}_t^T}{\widetilde{K}_t^T}\right)^{\frac{1}{\rho}}\psi = r_t+\delta \tag{B.36}$$

$$Z_t\left(\frac{\widetilde{Y}_t^N}{\widetilde{L}_t^N}\right)^{\frac{1}{\rho}}(1-\psi) = w_t \tag{B.37}$$

$$Z_t\left(\frac{\widetilde{Y}_t^N}{\widetilde{K}_t^N}\right)^{\frac{1}{\rho}}\psi = r_t+\delta \tag{B.38}$$

They imply:

$$\left(\frac{\widetilde{K}_t^T}{\widetilde{L}_t^T}\right)^{\frac{1}{\rho}} = \frac{w_t}{r_t+\delta}\frac{\psi}{1-\psi} \tag{B.39}$$

$$\left(\frac{\widetilde{K}_t^N}{\widetilde{L}_t^N}\right)^{\frac{1}{\rho}} = \frac{w_t}{r_t+\delta}\frac{\psi}{1-\psi} \tag{B.40}$$

which in turn imply that the capital labor ratios in the two sectors are the same:

$$\frac{\widetilde{K}_t^T}{\widetilde{L}_t^T} = \frac{\widetilde{K}_t^N}{\widetilde{L}_t^N}$$

From the household's choice with wage equalization (and clearing in the labor market) it still holds:

$$\widetilde{L}_t^T = \chi_t, \quad \widetilde{L}_t^N = 1-\chi_t$$

Therefore, $\widetilde{K}_t^T = (\chi_t/(1-\chi_t))\widetilde{K}_t^N$ which implies, from the clearing on the capital market ($\widetilde{K}_{t-1} = \widetilde{K}_t^T + \widetilde{K}_t^N$):

$$\widetilde{K}_t^T = \chi_t\widetilde{K}_{t-1}, \quad \widetilde{K}_t^N = (1-\chi_t)\widetilde{K}_{t-1}$$

Hence, from (B.39), it follows:

$$\widetilde{K}_{t-1} = \left(\frac{w_t\psi}{(r_t+\delta)(1-\psi)}\right)^{\rho} \tag{B.41}$$

With these results, the production function in the T-sector is:

$$\begin{aligned}\widetilde{Y}_t^T &= \left[\psi(\chi_t\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}} + (1-\psi)(\chi_t)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}} \\ &= \chi_t\left[\psi(\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}} + (1-\psi)\right]^{\frac{\rho}{\rho-1}}\end{aligned} \tag{B.42}$$

where the last expression has been obtained by multiplying and dividing by χ_t. By symmetry:

$$\widetilde{Y}_t^N = (1-\chi_t)\left[\psi(\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}} + (1-\psi)\right]^{\frac{\rho}{\rho-1}} \tag{B.43}$$

Plug (B.42) into (B.35) to have:

$$w_t = (1-\psi)\left[\psi(\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}} + (1-\psi)\right]^{\frac{1}{\rho-1}} \tag{B.44}$$

Plug (B.44) and (B.43) into (B.37) to have:

$$Z_t = 1 \tag{B.45}$$

which is unsurprising given that the analysis abstract from technological change and there is wage equalization between sectors. From (B.42), (B.43) and (B.45) output (per unit of labor efficiency and in terms of T-goods) is:

$$\widetilde{Y}_t \equiv \widetilde{Y}_t^T + Z_t\widetilde{Y}_t^N = \left[\psi(\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}} + (1-\psi)\right]^{\frac{\rho}{\rho-1}} \tag{B.46}$$

that is:

$$\widetilde{Y}_t^T = \chi_t\widetilde{Y}_t, \quad \widetilde{Y}_t^N = (1-\chi_t)\widetilde{Y}_t$$

On the household's side, nothing has changed as compared with the baseline specification. Hence, (B.14) holds true. With (B.45), it becomes:

$$(1+r_{t+1})\beta\frac{\zeta_{t+1}^g(N_{t+1}^g)^\sigma}{(L_{t+1}^g)^\sigma} = \left(\frac{\widetilde{C}_{t+1}^N}{\widetilde{C}_t^N}\right)^\sigma \tag{B.47}$$

Using the clearing condition in the N-sector: $\widetilde{C}_t^N = \widetilde{Y}_t^N = (1-\chi_t)\widetilde{Y}_t$, it becomes:

$$r_{t+1} = \frac{(L_{t+1}^g)^\sigma}{\beta\zeta_{t+1}^g(N_{t+1}^g)^\sigma}\left(\frac{(1-\chi_{t+1})\widetilde{Y}_{t+1}}{(1-\chi_t)\widetilde{Y}_t}\right)^\sigma - 1 \tag{B.48}$$

To close the model, one needs to find an expression for χ_t using the clearing condition in the market for T-goods:

$$\widetilde{C}_t^T + \widetilde{K}_tL_{t+1}^g = (1-\delta)\widetilde{K}_{t-1} + \widetilde{Y}_t^T$$

with $Z_t = 1$, $\widetilde{C}_t^N = \widetilde{Y}_t^N = (1-\chi_t)\widetilde{Y}_t$, from the household's first order condition:

$$\widetilde{C}_t^T = \frac{\gamma}{1-\gamma}(1-\chi_t)\widetilde{Y}_t$$

with $\widetilde{Y}_t^T = \chi_t\widetilde{Y}_t$, the clearing condition in the T-sector reads:

$$\frac{\gamma}{1-\gamma}(1-\chi_t)\widetilde{Y}_t + \widetilde{K}_tL_{t+1}^g = (1-\delta)\widetilde{K}_{t-1} + \chi_t\widetilde{Y}_t$$

that, once it is properly managed, gives:

$$\chi_t = \gamma + (1-\gamma)\frac{\widetilde{K}_tL_{t+1}^g - (1-\delta)\widetilde{K}_{t-1}}{\widetilde{Y}_t} \tag{B.49}$$

In sum, equations (B.44), (B.41), (B.49), (B.48), (B.46) characterize the dynamic equilibrium of the model when the production function is CES in both sectors:

$$
\begin{aligned}
w_t &= (1-\psi)(\widetilde{Y}_t)^{\frac{1}{\rho}} \\
\widetilde{K}_{t-1} &= \left(\frac{w_t\psi}{(r_t+\delta)(1-\psi)}\right)^{\rho} \\
\chi_t &= \gamma+(1-\gamma)\frac{\widetilde{K}_t L^g_{t+1}-(1-\delta)\widetilde{K}_{t-1}}{\widetilde{Y}_t} \\
r_{t+1} &= \frac{(L^g_{t+1})^{\sigma}}{\beta\zeta^g_{t+1}(N^g_{t+1})^{\sigma}}\left(\frac{(1-\chi_{t+1})\widetilde{Y}_{t+1}}{(1-\chi_t)\widetilde{Y}_t}\right)^{\sigma}-1 \\
\widetilde{Y}_t &= \left[\psi(\widetilde{K}_{t-1})^{\frac{\rho-1}{\rho}}+(1-\psi)\right]^{\frac{\rho}{\rho-1}}
\end{aligned}
$$

□

B.4 Endogenous labor supply

Household. Consider the representative household's problem (A.13) and evaluate variables in terms of efficiency units of exogenous labor ($L_t=\sum_j h_j N_{t,j}$). Again, for each choice variable X_t have $\widetilde{X}_t \equiv X_t/L_t$. The Lagrangian of the household's maximization problem with choice variables $\{\widetilde{C}^T_t,\widetilde{C}^N_t,\widetilde{K}_t,\widetilde{l}_t\}$ reads:

$$
\sum_{t=0}^{\infty}\beta^t\zeta_t\left[\frac{N^{\sigma}_t}{L^{\sigma-1}_t}\frac{(\widetilde{C}^T_t)^{\gamma(1-\sigma)}(\widetilde{C}^N_t)^{(1-\gamma)(1-\sigma)}}{1-\sigma}-\widetilde{\nu}_t\frac{(\widetilde{l}_t)^{1+\phi}}{1+\phi}\right]-\sum_{t=0}^{\infty}\beta^t\zeta_t\lambda_t\left\{\widetilde{C}^T_t+Z_t\widetilde{C}^N_t+\widetilde{K}_tL^g_{t+1}-\left[(1-\tau_t)w_t\widetilde{l}_t+(1+r_r)\widetilde{K}_t+\widetilde{T}_t\right]\right\}
$$

The first order conditions of this problem read:

$$
\begin{aligned}
\widetilde{C}^T_t &: \gamma\frac{(\widetilde{C}^T_t)^{\gamma(1-\sigma)}(\widetilde{C}^N_t)^{(1-\gamma)(1-\sigma)}}{\widetilde{C}^T_t}=\lambda_t\frac{L^{\sigma-1}_t}{N^{\sigma}_t} \\
\widetilde{C}^N_t &: (1-\gamma)\frac{(\widetilde{C}^T_t)^{\gamma(1-\sigma)}(\widetilde{C}^N_t)^{(1-\gamma)(1-\sigma)}}{\widetilde{C}^N_t}=\lambda_t\frac{Z_tL^{\sigma-1}_t}{N^{\sigma}_t} \\
\widetilde{K}_t &: \lambda_tL^g_{t+1}=\zeta^g_{t+1}\beta(1+r_{t+1})\lambda_{t+1} \\
\widetilde{l}_t &: \widetilde{\nu}_t(\widetilde{l}_t)^{\phi}=\lambda_t(1-\tau_t)w_t
\end{aligned}
$$

which can be rewritten as:

$$\lambda_t = \gamma\frac{(\widetilde{C}^T_t)^{\gamma(1-\sigma)}(\widetilde{C}^N_t)^{(1-\gamma)(1-\sigma)}}{\widetilde{C}^T_t}\frac{N^{\sigma}_t}{L^{\sigma-1}_t} \tag{B.50}$$

$$\widetilde{C}^T_t = \frac{\gamma}{1-\gamma}Z_t\widetilde{C}^N_t \tag{B.51}$$

$$\zeta^g_{t+1}\beta(1+r_{t+1})\frac{(N^g_{t+1})^{\sigma}}{(L^g_{t+1})^{\sigma}} = \left(\frac{\widetilde{C}^N_t}{\widetilde{C}^N_{t+1}}\right)^{-\sigma}\left(\frac{Z_t}{Z_{t+1}}\right)^{\gamma(1-\sigma)-1} \tag{B.52}$$

$$\widetilde{\nu}_t(\widetilde{l}_t)^{\phi} = \frac{N^{\sigma}_t}{L^{\sigma-1}_t}\gamma\left(\frac{\gamma Z_t}{1-\gamma}\right)^{\gamma(1-\sigma)-1}(\widetilde{C}^N_t)^{-\sigma}(1-\tau_t)w_t \tag{B.53}$$

The representative household chooses also how much to work in each sector. Under the assumption of perfect mobility of labor ($l_t=l^T_t+l^N_t$, i.e. $\widetilde{l}_t=\widetilde{l}^T_t+\widetilde{l}^N_t$) with the notation used in the other sections (recall

$0 < \chi_t < 1$) the optimal choice gives:

$$\widetilde{l}_t^T = \chi_t \widetilde{l}_t \quad \text{(B.54)}$$
$$\widetilde{l}_t^N = (1-\chi_t)\widetilde{l}_t \quad \text{(B.55)}$$

Firms. Under the assumption of no exogenous technology parameters, the Cobb-Douglas production function in both sectors can be rewritten taking into account the endogenous labor supply:

$$\widetilde{Y}_t^T = (\widetilde{K}_t^T)^{\psi}(\widetilde{l}_t^T)^{1-\psi}, \quad \widetilde{Y}_t^N = (\widetilde{K}_t^N)^{\psi}(\widetilde{l}_t^N)^{1-\psi}$$

Under the assumption of perfect competition, firms in each sector solve:

$$\max_{\widetilde{K}_t^T, \widetilde{l}_t^T} \widetilde{Y}_t^T - w_t \widetilde{l}_t^T - (r_t + \delta)\widetilde{K}_t^T, \quad \max_{\widetilde{K}_t^N, \widetilde{l}_t^N} Z_t \widetilde{Y}_t^N - w_t \widetilde{l}_t^N - (r_t + \delta)\widetilde{K}_t^N$$

which lead to the following first order conditions:

$$(1-\psi)\frac{\widetilde{Y}_t^T}{\widetilde{l}_t^T} = w_t$$
$$\psi\frac{\widetilde{Y}_t^T}{\widetilde{K}_t^T} = r_t + \delta$$
$$Z_t(1-\psi)\frac{\widetilde{Y}_t^N}{\widetilde{l}_t^N} = w_t$$
$$Z_t\psi\frac{\widetilde{Y}_t^N}{\widetilde{K}^N t} = r_t + \delta$$

It follows:

$$\frac{1-\psi}{\psi}\frac{\widetilde{K}_t^T}{\widetilde{l}_t^T} = \frac{w_t}{r_t+\delta}, \quad \frac{1-\psi}{\psi}\frac{\widetilde{K}_t^N}{\widetilde{l}_t^N} = \frac{w_t}{r_t+\delta} \quad \text{(B.56)}$$

that is, the capital labor ratio are the same in the two sectors ($\widetilde{K}_t^T/\widetilde{l}_t^T = \widetilde{K}_t^N/\widetilde{l}_t^N$). Therefore, using (B.54), (B.55) and the clearing condition in the capital market:

$$\widetilde{K}_t = \widetilde{K}_t^T + \widetilde{K}_t^N$$

it results:

$$\widetilde{K}_t^T = \chi_t \widetilde{K}_t \quad \text{(B.57)}$$
$$\widetilde{K}_t^N = (1-\chi_t)\widetilde{K}_t \quad \text{(B.58)}$$

From (B.56), it results:

$$\widetilde{K}_{t-1} = \frac{\psi}{1-\psi}\frac{w_t}{r_t+\delta}\widetilde{l}_t \quad \text{(B.59)}$$

while from the production functions:

$$\widetilde{Y}_t^T = \chi_t \widetilde{K}_{t-1}^{\psi}\widetilde{l}_t^{1-\psi}, \quad \widetilde{Y}_t^N = (1-\chi_t)\widetilde{K}_{t-1}^{\psi}\widetilde{l}_t^{1-\psi}$$

which plugged into the first order conditions above give:

$$w_t = (1-\psi)\left(\frac{\widetilde{K}_{t-1}}{\widetilde{l}_t}\right)^{\psi} \quad (B.60)$$

$$Z_t = 1 \quad (B.61)$$

Finally, aggregate output:

$$\widetilde{Y}_t \equiv \widetilde{Y}_t^T + Z_t \widetilde{Y}_t^N = \widetilde{K}_{t-1}^{\psi} \widetilde{l}_t^{1-\psi}$$

Goverment. As labor supply is an endogenous choice, contrary to the setting with exogenous labor supply, the government matters by setting the labor income tax rate τ_t. It is assumed that by running the pension system with fixed replacement rate $\bar{d}$, the government budget is balanced in each period (cf. equations (2.7) and (2.8) in the main text):

$$d_t = \bar{d} w_t (1-\tau_t)\bar{h}, \quad \tau_t w_t \widetilde{l}_t = d_t \frac{\sum_{j=jp}^{J} N_{t,j}}{L_t} = \widetilde{T}_t$$

which imply:

$$1-\tau_t = \frac{\widetilde{l}_t}{\widetilde{l}_t + \overline{dh}\widetilde{\Omega}_t} \quad (B.62)$$

where

$$\widetilde{\Omega}_t = \frac{\sum_{j=jp}^{J} N_{t,j}}{L_t} \quad (B.63)$$

is the *old-dependency ratio* in the model: the number of people to support with pension funds over the number of effective workers. Notice that the time of pension transfers jp is allowed to differ from the time of full retirement $jr+1$.

Clearing. The model is closed with the clearing conditions in the markets for the goods in the two sectors:

$$\widetilde{Y}_t^N = \widetilde{C}_t^N \quad (B.64)$$

$$\widetilde{Y}_t^T = \widetilde{C}_t^T + L_{t+1}^g \widetilde{K}_t - (1-\delta)\widetilde{K}_{t-1} \quad (B.65)$$

Using the clearing in the N-sector with the results above, it follows that $\widetilde{C}_t^T = \gamma/(1-\gamma)\widetilde{C}_t^N = \gamma/(1-\gamma)\widetilde{Y}_t^N = \gamma/(1-\gamma)(1-\chi_t)\widetilde{Y}_t$. Using this result and $\widetilde{Y}_t^T = \chi_t \widetilde{Y}_t$ into the clearing condition in the T-sector, it results (like in the previous sections):

$$\chi_t = \gamma + (1-\gamma)\frac{\widetilde{K}_t L_{t+1}^g - (1-\delta)\widetilde{K}_{t-1}}{\widetilde{Y}_t} \quad (B.66)$$

i.e. $1-\chi_t = (1-\gamma)(1-\iota_t)$, where ι_t identifies the investment rate $\iota_t \equiv (\widetilde{K}_t L_{t+1}^g - (1-\delta)\widetilde{K}_{t-1})/\widetilde{Y}_t$. Therefore, using the numbered equations above, the system of equations characterizing the dynamic equilibrium of the model with endogenous labor is the following:

$$
\begin{aligned}
1+r_{t+1} &= \frac{(L^g_{t+1})^\sigma}{\beta\zeta^g_{t+1}(N^g_{t+1})^\sigma}\left(\frac{\widetilde{Y}^N_t}{\widetilde{Y}^N_{t+1}}\right)^{-\sigma} \\
\widetilde{l}_t &= \left[\frac{N^\sigma_t}{\widetilde{\nu}_t L^{\sigma-1}_t}\gamma\left(\frac{\gamma}{1-\gamma}\right)^{\gamma(1-\sigma)-1}(\widetilde{Y}^N_t)^{-\sigma}(1-\tau_t)w_t\right]^{\frac{1}{\phi}} \\
1-\tau_t &= \frac{\widetilde{l}_t}{\widetilde{l}_t+\overline{dh}\widetilde{\Omega}_t} \\
\widetilde{Y}_t &= \widetilde{K}^\psi_{t-1}\widetilde{l}^{1-\psi}_t \\
w_t &= (1-\psi)\left(\frac{\widetilde{K}_{t-1}}{\widetilde{l}_t}\right)^\psi \\
\widetilde{K}_{t-1} &= \widetilde{l}_t\frac{\psi w_t}{(1-\psi)(r_t+\delta)} \\
\widetilde{Y}^N_t &= (1-\chi_t)\widetilde{Y}_t \\
\chi_t &= \gamma+(1-\gamma)\frac{\widetilde{K}_tL^g_{t+1}-(1-\delta)\widetilde{K}_{t-1}}{\widetilde{Y}_t}
\end{aligned}
$$

with endogenous variables $\{r_t,\widetilde{Y}^N_t,\widetilde{Y}_t,\widetilde{l}_t,\widetilde{K}_t,w_t,\tau_t,\chi_t\}^\infty_{t=0}$. Equivalently, it can be rewritten as:

$$
\begin{aligned}
1+r_{t+1} &= \frac{(L^g_{t+1})^\sigma}{\beta\zeta^g_{t+1}(N^g_{t+1})^\sigma}\left(\frac{(1-\iota_{t+1})\widetilde{Y}_{t+1}}{(1-\iota_t)\widetilde{Y}_t}\right)^{\sigma} \\
\widetilde{l}_t &= \left[\frac{N^\sigma_t}{\widetilde{\nu}_t L^{\sigma-1}_t}\gamma\left(\frac{\gamma}{1-\gamma}\right)^{\gamma(1-\sigma)-1}\left((1-\gamma)(1-\iota_t)\widetilde{Y}_t\right)^{-\sigma}(1-\tau_t)w_t\right]^{\frac{1}{\phi}} \\
1-\tau_t &= \frac{\widetilde{l}_t}{\widetilde{l}_t+\overline{dh}\widetilde{\Omega}_t} \\
\iota_t &= \frac{(\widetilde{K}_tL^g_{t+1}-(1-\delta)\widetilde{K}_{t-1})}{\widetilde{Y}_t} \\
\widetilde{Y}_t &= \widetilde{K}^\psi_{t-1}\widetilde{l}^{1-\psi}_t \\
w_t &= (1-\psi)\left(\frac{\widetilde{K}_{t-1}}{\widetilde{l}_t}\right)^\psi \\
\widetilde{K}_{t-1} &= \widetilde{l}_t\left(\frac{}{r_t+\delta}\right)^{\frac{1}{1-\psi}}
\end{aligned}
$$

where the endogenous variables are $\{r_t,w_t\widetilde{Y}_t,\widetilde{l}_t,\widetilde{K}_t,\tau_t,\iota_t\}^\infty_{t=0}$ and the exogenous variables (depending uniquely on demographics) are for all periods t:

$$
N_t=\sum_{j=0}^{J}N_{t,j},\quad L_t=\sum_{j=0}^{jr}h_jN_{t,j},\quad \zeta_t=\sum_{j=0}^{J}\pi_{t,j}\left(\frac{N_{t,j}}{N_t}\right),\quad \widetilde{\Omega}_t=\sum_{j=jr+1}^{J}\frac{N_{t,j}}{L_t},\quad \widetilde{\nu}_t=\left[\sum_j N_{t,j}(\widetilde{h}_j)^{1+\frac{1}{\phi}}(\nu_j)^{-\frac{1}{\phi}}\right]^{-\phi}
$$

with $N^g_{t+1}=L_{t+1}/L_t, L^g_{t+1}=L_{t+1}/L_t, \zeta^g_{t+1}=\zeta_{t+1}/\zeta_t$. Compared to the case of exogenous labor supply, as exogenous variables there are also the old-dependency ratio $\widetilde{\Omega}_t$ and the wedge to the aggregate disutility of labor $\widetilde{\nu}_t$. □

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