Money and Spending Multipliers with HA-IO

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This paper derives money and spending multipliers in an economy with multiple agents and multiple sectors of production, arranged in an input-output network. Different agents consume different bundles of goods, and different sectors hire different bundles of agents and fixed factors. As a result, agents face heterogeneous nominal and real rigidities, in the form of (i) different wage rigidity, or different price rigidity in the sectors where they work, etc.; (ii) different labor supply elasticity; (iii) different complementarity with fixed factors. In the cross-section, a monetary expansion reallocates employment towards workers who face stronger nominal rigidities and weaker real rigidities, because the relative price of their products falls. In the aggregate, the ability to substitute towards cheaper goods increases monetary non-neutrality. Similarly, the spending multiplier is larger when spending increases the demand for workers facing more nominal and less real rigidities. In a generic economy, the multiplier has the same form as in a representative agent economy if and only if spending replicates the aggregate consumption basket. The composition of spending is irrelevant for aggregate employment only in economies where all workers face the same pricing frictions, labor supply elasticity and exposure to fixed factors.

Preliminary and incomplete

1 Introduction

How do economies respond to monetary policy and government spending? Traditionally these questions have been approached from an aggregate point of view, in an economy populated by a representative agent consuming a single good. This prevents from studying whether and how monetary policy redistributes employment and income across different workers, and whether worker heterogeneity affects the response of macroeconomic aggregates to monetary policy and government spending.

This paper revisits money and spending multipliers in a realistic economy, with many agents and many sectors, arranged in an input-output network. Agents consume different bundles of goods and supply different kinds of labor, while sectors hire different bundles of workers and fixed factors.[1] Labor and product markets are subject to heterogeneous pricing frictions, and agents have different labor supply elasticity. As a result, agents face different degrees of nominal and real rigidities.

This is the key channel through which monetary policy redistributes income and employment across different workers.[2] A monetary expansion redistributes employment towards agents who face more nominal rigidities (because they have stickier wages, or are employed by sectors with stickier price, etc.) and/or less real rigidities (i.e. they have more elastic labor supply, or are employed by sectors that rely less on fixed factors). This is because the relative price of their products falls, thereby raising demand for their labor. In the aggregate, the ability of consumers and producers to substitute towards cheaper goods increases the response of employment to a monetary expansion.

Intuitively, when workers have different wage rigidity, or are employed in production chains with different overall price rigidity, in response to a monetary expansion producers and consumers substitute towards sticky products or factors. These factors have a flatter Phillips curve, which increases the money multiplier. The same intuition carries through when workers have heterogeneous labor supply elasticities. The vice versa holds after a monetary contraction. The model thus formalizes the common wisdom that regions in a currency union with more frictional labor markets suffer more severely from contractionary monetary policy.

The role of fixed factors is more nuanced, and depends on the interaction between producers' substitution across labor and fixed factors and consumers' substitution across goods. If labor and fixed factors are strong complements, while consumers can freely substitute, then a monetary expansion favors sectors which rely less on fixed factors, and the workers whom they employ. By contrast, if producers can substitute between labor and fixed factors, while consumption goods are strong complements, then a monetary expansion favors the workers who rely more on fixed factors, by increasing their labor share. To make this discussion more concrete, consider two housing markets which differ in land availability. If people are unwillingly to migrate, the inhabitants of the land-scarce city prefer to improve their small apartments instead of moving to the land abundant city, and employment raises by more for

[1]Depending on the question of interest, different agent types can be viewed as individuals working in different occupations or industries within the same region, or as individuals living in different regions of a currency union. When interpreted in this way, the paper extends traditional currency union models (such as Gali and Monacelli (2008)) to allow for a quantitatively realistic input-output and preference structure.

[2]Different from the HANK literature (Kaplan et al. (2018)), we abstract from heterogeneity in marginal propensities to consume.

construction workers in this market (who had a smaller share to begin with). Vice versa, the land abundant market gains when people are willing to migrate.

Following a similar logic as with the money multiplier, the spending multiplier is larger when spending increases the demand for workers who face stronger nominal and/or weaker real rigidities. In a generic economy, the spending multiplier is the same function of the money multiplier and aggregate spending as in a representative agent economy if and only if spending replicates the aggregate consumption basket, thereby leaving the relative demand for different workers unaffected. By contrast, the composition of spending is irrelevant for aggregate employment only when all workers face the same nominal and real rigidities. Two important instances of this are economies with identical preferences and a symmetric input-output network, or with identical preferences, flexible prices and no fixed factors.

Formally, money and spending multipliers are obtained by intersecting the supply and demand blocks of the model. The supply block captures the workers' consumption-leisure tradeoff, and the transmission of wages into prices. This is similar to a representative agent economy, with the caveat that we now have one consumption-leisure tradeoff per each worker, and changes in one worker's wage can affect other workers as well.

The demand block is composed of two parts: aggregate expenditure, which is common to the representative agent economy, and relative factor demand, which is novel. A monetary shock only affects aggregate expenditure, while a change in government spending also alters the relative demand for different factors. Nonetheless, monetary shocks have an impact on relative prices in the presence of heterogeneous nominal and/or real rigidities. This is because even a proportional increase in all factor prices can break the factor market equilibrium. Intuitively, demand increases for workers with stickier wages, or who are employed by stickier sectors (etc.), because their products become relatively cheaper. By contrast, supply increases for workers with higher Frish elasticity. Moreover, the relative price of goods which rely less on fixed factors falls, thereby increasing demand for the factors which produce them. Thus, after a monetary expansion or a change in government spending, relative factor prices must adjust to rebalance the markets.

This effect is novel to the heterogeneous agent economy. It disappears only if all workers face the same nominal and real rigidities, or monetary policy and government spending are such that relative factor demand is unaffected. This is the case only when government spending replicates the aggregate consumption basket, and monetary policy adjusts to keep private expenditure constant.

The theory highlights which dimensions of heterogeneity across workers and sectors can have important quantitative implications for the aggregate and redistributive effects of policy. The calibration is still work in progress.

1.1 Related Literature

The framework in this paper is similar to Baqaee and Farhi (2018), who study the propagation and aggregation of productivity and markup shocks in an economy with multiple agents and a general input-output network. While markups are exogenous in Baqaee and Farhi (2018), here they arise endogenously in response to policy changes under sticky prices. Moreover, Baqaee and Farhi (2018) focus on sector-level output and prices, whereas this paper focuses on agent-level employment and income.

A large literature studies New Keynesian models of currency unions (Aoki (2001); Benigno (2004), Devereux and Engel (2003), Huang and Liu (2007)). Gali and Monacelli (2008) specifically focus on fiscal policy, showing that it can achieve better outcomes than monetary policy alone in the presence of fixed exchange rates. This paper complements previous works by introducing a quantitatively realistic representation of production.

A recent literature also discusses monetary non-neutrality in representative-agent production networks (Carvalho (2006), Nakamura and Steinsson (2010), Pasten et al. (2019), LaO and Tahbaz-Salehi (2019), Rubbo (2020)). Most closely related to this paper are Flynn et al. (2021), Bouakez et al. (2020) and Cox et al. (2020), who study spending multipliers in multi-sector economies. While Patterson et al. consider heterogeneity in marginal propensities to consume, this paper is about heterogeneity in the nominal and real rigidities faced by different agents. This is also the focus of Bouakez et al. (2020) and Cox et al. (2020). In particular, Cox et al. (2020) also show that spending on sticky-price sectors increases the aggregate multiplier. Their approach however is mainly empirical, whereas this paper derives a general theory.

2 Environment

This section lays out the assumptions about preferences, production and policy instruments, and derives optimality conditions for consumers and producers. Section 2.4 introduces the equilibrium concept, which is designed to account for the endogenous evolution of sectoral markups under price rigidities.

2.1 Consumers

There are H agent types, which correspond to the H labor types in the economy. The preferences of type-h individuals are described by the utility function

$$U = \frac{C_h^{1-\gamma_h}}{1-\gamma_h} - \frac{L_h^{1+\varphi_h}}{1+\varphi_h} \tag{1}$$

Agents enjoy consumption (C) and dislike labor (L). We allow for agents to have different wealth effects γ_h and inverse Frish elasticities φ_h.

There are N goods produced in the economy, and consumers have homothetic preferences over all of these goods. Agents of different type consume these goods in potentially different proportions. I denote consumption utility by $C_h \equiv C_h(c_1, ..., c_N)$, defined over bundles $(c_1, ..., c_N)$.

There are also J fixed factors, $K_1, ..., K_J$. We denote the share of factor j owned by type-h agents by ζ_{jh}, and we collect factor ownership shares into the matrix $\mathcal{Z}$.

Agents own shares of firms in each sector as well. We denote by ϑ_{ih} the share of firms in sector i owned by type-h agents and we collect sector ownership shares into the matrix Θ.

Consumers are subject to the budget constraint

$$P_h^c C_h \leq W_h L_h + \sum_j \zeta_{jh} R_j K_j + \sum_i \vartheta_{ih} \Pi_i - T_h \tag{2}$$

where P_c is the price index implied by the consumption aggregator C, W_h is the nominal wage of agent h, Π is the vector of firm profits (sector-by-sector), which are rebated lump-sum to households proportionately to their ownership shares of firms, and T_h is a lump-sum tax or transfer received by type h.

It is convenient to decompose the lump-sum taxes $\{T_h\}_{h=1}^{H}$ into seignorage revenues, taxes to finance government spending, and taxes to finance input subsidies. I assume that seignorage revenues are distributed to agents in proportion to their consumption shares, so that type-h agents receive an amount $s_h d \log M$. This is the same amount that they need to spend to purchase money for consumption, so that seignorage and money purchases cancel out from the agents' budget constraint. I also assume that each agent pays for input subsidies proportionately to his ownership share in the sectors that receive them. Finally, to finance government spending agents are taxed proportionately to the increase in demand for the labor and fixed factors that they own.

Consumers maximize utility (1) subject to the budget constraint (2).

The optimal consumption-leisure tradeoff satisfies the first order condition

$$\frac{W_h}{P_{ch}} = C_h^{\gamma_h} L_h^{\varphi_h} \tag{3}$$

2.2 Producers

There are N sectors in the economy (indexed by $i \in \{1, ..., N\}$). Within each sector there is a continuum of firms, producing differentiated varieties.

All firms f in sector i have the same constant returns to scale production function

$$Y_{if} = A_i F_i(\{L_{ihf}\}, \{K_{ijf}\}, \{x_{itf}\})$$

where L_{ihf} is the amount of type-h labor hired by firm f in sector i, K_{ijf} is the amount of fixed factor j, x_{itf} is the quantity of good t that it uses as input, and A_i is a Hicks-neutral, sector-specific TFP shifter. Labor is freely mobile across sectors (but not across types).

Customers (consumers and other producers) buy a CES bundle of sectoral varieties, with elasticity of substitution ϵ_i. Sectoral output is given by

$$Y_i = \left(\int Y_{if}^{\frac{\epsilon_i - 1}{\epsilon_i}} df \right)^{\frac{\epsilon_i}{\epsilon_i - 1}}$$

and the implied sectoral price index is

$$P_i = \left(\int p_{if}^{1-\epsilon_i} df \right)^{\frac{1}{1-\epsilon_i}}$$

The government provides input subsidies to offset the markup distortions arising from monopolistic competition. Sectoral subsidies are constant over time, and given by

$$1 - \tau_i = \frac{\epsilon_i^* - 1}{\epsilon_i^*}$$

All producers in every sector i minimize costs given factor and input prices. With constant returns to scale marginal costs are the same for all firms, and all firms use inputs in the same proportions. Sector i's marginal cost (MC_i) is the solution of the following problem:

$$MC_i = min_{\{x_{it}\},\{L_{ih}\},\{K_{ij}\}} \sum_h W_h L_{ih} + \sum_j R_j K_{ij} + \sum_t P_t x_{it} \quad s.t. \; A_i F_i \left(\{L_{ih}\}, \{K_{ij}\}, \{x_{it}\} \right) = 1 \tag{4}$$

Prices are set à la Calvo: only a fraction δ_i of the firms in each sector i can update their price in a given period. Adjusting firms set prices to maximize profits, and set their price equal to the pre-subsidy marginal cost:

$$P_i^* = MC_i \tag{5}$$

Non-adjusting firms instead absorbs all cost changes in their markup.

Remark 1. In this framework, wage rigidity can be represented in the same way as price rigidity. This is done by adding H labor unions to the network, which hire workers and sell their services to all other sectors. Wage rigidity can be represented as price rigidity in the labor unions.[3] Therefore theoretical analysis will not explicitly distinguish between price and wage rigidity.

2.3 Timing and policy instruments

The central bank decides the money supply (i.e. it controls aggregate nominal GDP) and the amount of government spending on each sector, denoted by the $N \times 1$ vector $\{G_i\}$. It also levies lump-sum taxes T_h on the various agents to balance the government budget (i.e. to cover seignorage, spending and input subsidies).

The economy is static, and subject to an aggregate cash-in-advance constraint on consumption expenditures:

$$\sum_h P_{ch} C_h + \sum_i G_i \leq M \tag{6}$$

M is the aggregate money supply.

[3]While the market wage, defined as the price required by the labor sector, is sticky, the underlying wage paid by the labor sector to workers is always assumed to be flexible.

Prices are set at a value p_{-1} before the central bank decides the money supply and government spending. After the decision is made, only a subset of firms in each sector can update their price based on the new information, according to the sectoral Calvo parameters.

2.4 Equilibrium

The equilibrium concept adapts the definition in Baqaee and Farhi (2020) to account for the endogenous determination of markups given pricing frictions and shocks. As in Baqaee and Farhi (2020), markets must clear at the given sectoral markups; we then further require the evolution of markups to be consistent with Calvo pricing and the realization of policy.

For given sectoral probabilities of price adjustment δ_i, sectoral productivity shifters A_i and money supply M, general equilibrium is given by a vector of firm-level markups $\mathcal{M}_{fi}$, a vector of sectoral prices P_i, vectors of nominal wages W, returns to fixed factors R, labor supplies L, sectoral outputs Y_i, a matrix of intermediate input quantities x_{ijt}, and a matrix of final demands c_{ik} such that: a fraction δ_i of firms in each sector i charges the profit-maximizing price given by (5); the markup charged by adjusting firms is given by the ratio of the profit-maximizing price and marginal costs, while the markups of non-adjusting firms are such that their price remains constant; consumers maximize utility subject to the budget constraint; producers in each sector i minimize costs and charge the relevant markup; and markets for all goods and factors clear.

Remark 2. Note that this equilibrium concept nests the standard one with flexible prices, which is obtained as a special case when $\delta_i = 1$ for every sector i.

3 Log-linearized model

The theoretical analysis is based on a log-linear approximation of the model around the efficient equilibrium with flexible prices, no government spending, and unit nominal GDP.

Sections 3.1 and 3.3 introduce a set of variables and parameters which fully characterize the approximated model. This notation will be useful in Section 4, which derives the supply and demand blocks of the model. Section 5 then solves for the equilibrium to derive money and spending multipliers.

3.1 Variables

Table 1 introduces the variables that characterize the supply and demand blocks of the model. Lower case letters denote logs.

Employment	l$=\begin{pmatrix} l_1 & ... & l_H \end{pmatrix}^T$
Sectoral inflation rates	$\pi = \begin{pmatrix} \pi_1 & ... & \pi_N \end{pmatrix}^T, \ \pi_i \equiv p_i - p_{i,-1}$
Sectoral markups	$\mu = \begin{pmatrix} \mu_1 & ... & \mu_N \end{pmatrix}^T, \ \mu_i \equiv p_i - mc_i$

Table 1: Model variables

Remark 3. Table 1 does not define employment at the sector level, but at the agent level. This is because the analysis of money and spending multipliers will focus on agent-level outcomes, rather than sector-level.

3.2 Agents' parameters

The consumers' side of the model is summarized by their preferences, and their budget constraint. Table 2 introduces the parameters which characterize them in the log-linearized model.

Consumption shares	$\beta \in \mathbb{R}^{N,H}, \ \beta_{ih} = \frac{P_i C_{ih}}{GDP}$
Wealth effects	$\Gamma \in \mathbb{R}^H, \ \Gamma \equiv diag\left(\gamma_1, ..., \gamma_H\right)$
Inverse Frish elasticities	$\Phi \in \mathbb{R}^H, \ \Phi \equiv diag\left(\varphi_1, ..., \varphi_H\right)$
Factor ownership	$\mathcal{Z} \in \mathbb{R}^{J,H}, \ \zeta_{jh} \equiv \frac{K_{jh}}{K_j}$
Sector ownership	$\Theta \in \mathbb{R}^{N,H}, \ \zeta_{ih} \equiv \frac{\Pi_{ih}}{\Pi_i}$
Factor shares	$\varsigma_h = \frac{W_h L_h}{GDP}, \ \varsigma_j = \frac{R_j K_j}{GDP}$ $\mathcal{S} = diag\left(\varsigma\right)$
Income shares	$S = diag\left(s_1, ..., s_H\right), \ s_h = \frac{P_{ch} C_h}{GDP}$
Labor shares	$\Lambda \in \mathbb{R}^H, \ \Lambda_h = \frac{W_h L_h}{P_h^C C_h}$

Table 2: Input-output definitions

3.3 Production parameters

3.3.1 Price rigidity

I collect sectoral probabilities of price adjustment $\{\delta_i\}_{i=1}^N$ into the diagonal matrix Δ.

3.3.2 Input-output definitions

Up to the first order, the production structure is fully characterized by the equilibrium factor and input shares, and by the relevant Morishima elasticities of substitution. Table 3 introduces notation for these parameters. It also defines the Leontief inverse, the Domar weights, and the factor content matrix, which are derived from the input-output matrix and the vector of consumption shares.

Labor shares	$\alpha \in \mathbb{R}^N,\ \alpha_i = \frac{WL_i}{MC_i Y_i}$
Input-output matrix	$\Omega \in \mathbb{R}^N \times \mathbb{R}^N,\ \omega_{ij} = \frac{P_j x_{ij}}{MC_i Y_i}$
Leontief inverse	$(I-\Omega)^{-1}$
Domar weights	$\lambda^T = \beta^T (I-\Omega)^{-1}$
Factor contents	$\begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \alpha^T \lambda$
Substitution elasticities	$\left\{\sigma^C_{ij}\right\}$ consumption $\left\{\theta^i_{jk}, \theta^i_{jL}\right\}$ production ϵ_i between varieties

Table 3: Input-output definitions

The input-output matrix captures the direct exposure of each sector i to every other sector j. The Leontief inverse instead captures the total exposure, either directly or indirectly through intermediate inputs. To see this, one can write an expansion of the Leontief inverse where the n-th term is the exposure of i to j through paths of length n:

$$(I-\Omega)^{-1}_{ij} = I_{ij} + \Omega_{ij} + \Omega^2_{ij} + ...$$

Remark 4. Factors and input shares must sum to one: $(\Omega + \alpha)\,\mathbf{1} = \mathbf{1}$. This implies the following relations:

$$(I-\Omega)^{-1} \alpha \mathbf{1} = \mathbf{1},\ \lambda^T \alpha \mathbf{1} = 1$$

Remark 4 states that the total (direct and indirect) share of the primary factors in each sector's cost is one. This is intuitive, because with constant returns to scale and marginal cost pricing (as guaranteed by the optimal subsidies introduced in Section 2.2) all value added comes from primary factors. Therefore the share of primary factors in aggregate value added ($\lambda^T \alpha \mathbf{1}$) is also one.

The factor content matrix decomposes the share of each factor into each agent's final consumption. The element (h, l) corresponds to the total share of factors owned by agent h in agent l's consumption. It is immediate to verify that $\mathbf{1}^T \begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \alpha^T \lambda = \mathbf{1}^T$, that is, the total share of primary factors in each agent's consumption is one.

Assumption 1. *To ensure that all relative prices are well defined, we impose that*

$$rank\left(I_H - \begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \alpha^T \lambda\right) = H - 1$$

Assumption 1 is equivalent to asking that, for each agent pair (h, l), there is a path $(h, h_1, ..., h_T, l)$ connecting h to l. Linkages on this path correspond to agent h_t consuming a good produced using labor from agent h_{t-1}, or a fixed factor owned by this agent.

Lemmas 1 and 2 below relate factor and income shares in the flex-price equilibrium with the producers' demand for factors and intermediate inputs, and the consumers' demand for final goods.

Lemma 1. *Factor and agents' shares are tied by the relation*

$$\varsigma = \alpha^T \lambda s$$

Lemma 1 tells us that each factor's share in total GDP is given by the sum of its shares in each agent h's consumption, $\left(\alpha^T \lambda\right)_h$, weighted by the agent's income share s_h. Lemma 2 instead states that each agent's income share s_h is equal to the sum of the income shares of the factors that he owns.

Lemma 2. *Under Assumption 1 , the vector s of agents' income shares is the unique solution of the equation*

$$\left(I_H - \begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \alpha^T \lambda\right) s = \mathbf{0}$$

4 Supply and demand

This Section derives the supply and demand blocks of the model. The supply block combines the evolution of marginal costs, prices and markups (as a function of factor prices) with the labor supply curve. The demand block instead characterizes the evolution of aggregate GDP and factor income shares. The supply block and aggregate GDP are similar to representative agent models, and, to the first order, are the same as in a Cobb-Douglas economy. In multi-factor models one also needs to track the evolution of their income shares. This crucially depends on substitution elasticities in production and consumption, which govern the degree of expenditure switching in response to changes in relative factor prices.

4.1 Supply block

Marginal costs, prices and markups Equation (7) log-linearizes the optimal reset price condition (5), and imposes that only a fraction Δ of producers can adjust their price:

$$\pi = \Delta \left(mc - \mathbb{E}mc\right) \tag{7}$$

In turn, sectoral marginal costs can be derived from the cost minimization problem (4):

$$mc - \mathbb{E}mc = \alpha \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \Omega\pi \tag{8}$$

Equation 8 tells us that sector i's marginal cost is exposed to factor prices π_w and π_r through its factor shares $\alpha_{(i)}$, and to input prices proportionately to its input shares $\Omega_{(i)}$. Combining equations (7) and (7) allows us to write sectoral inflation rates as a function of factor prices:

$$\pi = \Delta \left(I - \Omega\Delta\right)^{-1} \alpha \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} \tag{9}$$

The producers who cannot update their price must absorbs cost changes into their markup. As a result, the overall markup of sector i changes by

$$\mu_i = -\left(1 - \delta_i\right) mc_i \tag{10}$$

Equations (10) and (7) imply the following relationship between sectoral inflation rates and markups:

$$\mu = -\left(I - \Delta\right) \Delta^{-1} \pi \tag{11}$$

Budget constraint Log-linearizing the budget constraint (2) yields:

$$\pi^C + c = \underbrace{diag\left(\Lambda\right)\left(\pi_w + l\right)}_{\text{labor income}} + S^{-1} \left[\underbrace{\mathcal{Z}^T \mathcal{S}_K \pi_r}_{\text{income from fixed factors}} + \underbrace{\Theta^T diag \bar{\lambda} \mu}_{\text{profits net of subsidies}} - \underbrace{\begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \left[\left(I - \Omega\right)^{-1} \alpha\right]^T G}_{\text{spending tax}} \right] \tag{12}$$

where π^C is the vector of agent-level consumer price inflation:

$$\pi^C \equiv \beta^T \pi$$

Consumption-leisure tradeoff Log-linearizing the consumption-leisure tradeoff (3) yields

$$\Gamma c + \Phi l = \pi_w - \pi^C \tag{13}$$

Combining equation (13) with the consumers' budget constraint (12) and the evolution of prices (9) and markups (11) allows us to express labor supply as a function of factor prices:

$$l = \mathbb{S}_w \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \mathbb{S}_G G \tag{14}$$

where

$$\mathbb{S}_w \equiv \left[\Gamma diag\left(\Lambda\right) + \Phi\right]^{-1} \left(\begin{pmatrix} I - \Gamma diag\left(\Lambda\right) & -\Gamma S^{-1} \mathcal{Z} \mathcal{S}_K \end{pmatrix} - \left[\left(I - \Gamma\right) \beta^T \Delta - \Gamma S^{-1} \Theta^T diag \bar{\lambda} \left(I - \Delta\right)\right] \left(I - \Omega\Delta\right)^{-1} \alpha \right)$$

$$\mathbb{S}_G \equiv \left[\Gamma diag\left(\Lambda\right) + \Phi\right]^{-1} \Gamma S^{-1} \begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \left[\left(I-\Omega\right)^{-1}\alpha\right]^T$$

Equation (14) is the key relation that captures the supply block of the model. The next Section illustrates it in three special economies, to build intuition. In the heterogeneous agent, flex price economy there are spillover effects from nominal to real wages. The representative agent economy instead illustrates the positive correlation between nominal and real wages in the presence of sticky prices. The third example illustrates the role of fixed factors.

4.1.1 Examples

Example 1. Flex-price economy

Consider an economy with flexible prices and no fixed factors. We have

$$\mathbb{S}_w = \left[\Gamma + \Phi\right]^{-1} \left(I - \Gamma\right)\left(I - \lambda^T \alpha\right)$$

Here the matrix $I - \lambda^T \alpha$ maps changes in nominal wages into changes in real wages. With a representative agent, changes in nominal wages would be fully reflected into consumption prices ($\lambda^T \alpha = 1$, following Remark 4), thereby leaving the real wage unchanged. With multiple agents, instead, a change in the nominal wage of only one worker h is not fully reflected into the price of her consumption bundle. Instead, it leads to an increase in h's real wage proportional to the share $1 - \left(\lambda^T \alpha\right)_{hh}$ of other workers in her final consumption. For all other workers m real wages fall proportionately to the share of h in their final consumption, $\left(\lambda^T \alpha\right)_{mh}$.

Following the same intuition as in the representative agent economy, Remark 4 implies that when all nominal wages increase proportionately real wages remain unaffected ($\left(I - \lambda^T \alpha\right)\mathbf{1} = \mathbf{0}$). Similarly, Lemma 2 implies that aggregate real wages do not change ($s^T\left(I - \lambda^T \alpha\right) = \mathbf{0}^T$). This in turn implies that changes in nominal wages have no effect on aggregate employment, in an economy where all agents have the same wealth effects and Frish elasticities ($\Gamma = \gamma I$, $\Phi = \varphi I$). Intuitively, all workers respond to changes in their real wages with the same elasticity, and real wages do not change on average. Therefore the average labor supply does not change either.

Example 2. Representative agent economy

Consider an economy with a representative agent and no fixed factors. Equation (14) becomes

$$l = \frac{1}{\gamma + \varphi}\left[\left(1 - \delta_\beta\left(\alpha\right)\right)\pi_w + \gamma G\right]$$

where

$$\delta_\beta\left(\alpha\right) \equiv \beta^T \Delta \left(I - \Omega\Delta\right)^{-1} \alpha$$

is the pass-through of factor prices into consumer prices. In the presence of sticky prices we have $\delta_\beta\left(\alpha\right) < 1$. This means that changes in nominal wages affect real wages, because prices only partially adjust. Therefore an increase in nominal wages has a positive effect on labor supply.

The presence of government spending also increases labor supply, through a wealth effect. Workers are taxed to finance spending, which decreases their consumption and thereby makes them willing to supply more labor at a given wage.

Example 3. Fixed factors

Consider an economy with flexible prices, a representative agent, and one fixed factor, which we will call capital, with share $1-\alpha_L$. We have

$$\mathbb{S}_w \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = \frac{1-\alpha_L}{\alpha_L \gamma + \varphi} \left(\pi_w - \pi_r\right)$$

An increase in the nominal wage, for constant rental rate of capital, now increases real wages and labor supply. This happens because nominal wages impact consumer prices only proportionately to the labor share α_L. An increase in the rental rate of capital has the opposite effect. Following Remark 4, a proportional increase in both nominal wages and the rental rate of capital has no effect on real wages and labor supply.

4.2 Demand block

The demand block of the model is composed of two parts. The first is common with the representative agent economy, and captures the evolution of aggregate GDP. The second is novel to the multi-factor economy, and describes the evolution of factor income shares.

Aggregate GDP Aggregate GDP is managed by the central bank, through the cash-in-advance constraint (equation (6)). Log-linearizing this constraint, and substituting for consumption from the budget constraint (12), and for markups using equation (11), yields

$$\delta_{\bar{\beta}}\left(\alpha\right) \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \varsigma_L^T l = d \log M \tag{15}$$

Equation (15) tells us that the log-change in aggregate nominal expenditure is given by the change in aggregate output, which in turn is equal to the change in employment ($\varsigma_L^T l$), times the change in the aggregate consumer price index. The latter can be written as a function of factor prices using the pass-through operator $\delta_{\bar{\beta}}\left(\alpha\right)$.

Factor income shares The evolution of in factor income shares is given by

$$d\varsigma = d\alpha^T \lambda s + \alpha^T d\lambda s + \alpha^T \lambda ds + \left[\left(I-\Omega\right)^{-1} \alpha\right]^T \frac{G}{GDP} \tag{16}$$

The first free terms come from changes in private demand, while the last comes from spending. Changes in private demand in turn come from income redistribution across agents with different preferences (ds), changes in consumption and input shares ($d\lambda$), and changes in factor shares ($d\alpha$). The changes in factor, input and consumption shares crucially depend on the degree of expenditure switching by producers and consumers in response to changes in relative prices. They are characterized in Lemma 3 below.

Changes in the agents' income shares ds can be derived from their budget constraint (12):

$$ds = \Theta^T diag\bar{\lambda}\mu + \begin{pmatrix} I_H & \mathcal{Z} \end{pmatrix} \left[d\varsigma - \left[(I-\Omega)^{-1}\alpha \right]^T G \right] \tag{17}$$

where the first term corresponds to profit income, the second to income from labor and fixed factors, and the last from the spending tax.

We can then solve for the change in factor income shares directly from their definition, which yields:

$$d\varsigma = \mathcal{S} \left[\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \begin{pmatrix} l \\ \mathbf{0} \end{pmatrix} - \mathbf{1} d\log GDP \right] \tag{18}$$

Merging (19), (17) and (19) allows us to relate labor demand with factor prices and government spending:

$$\left(I - \alpha^T\lambda \begin{pmatrix} I_K & \mathcal{Z} \end{pmatrix}\right) \mathcal{S} \begin{pmatrix} l \\ \mathbf{0} \end{pmatrix} = -\left(I - \alpha^T\lambda \begin{pmatrix} I_K & \mathcal{Z} \end{pmatrix}\right) \mathcal{S} \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + d\alpha^T\lambda s + \alpha^T d\lambda s +$$

$$+\alpha^T\lambda\Theta^T diag\bar{\lambda}\mu + \left(I - \alpha^T\lambda \begin{pmatrix} I_K & \mathcal{Z} \end{pmatrix}\right) \left[(I-\Omega)^{-1}\alpha \right]^T G \tag{19}$$

Lemma 3 characterizes changes in consumption, input and factor shares as a function of markups and factor prices.

Lemma 3. *The response of consumption, input and factor shares to markups and factor prices is given by*

$$d\alpha^T\bar{\lambda} + \alpha^T d\lambda s = \left[\mathcal{S}\left(I - \mathbf{1}\varsigma^T\right) - Cov_s\left(\lambda^T\alpha\right) - \mathcal{Y}_w \right] \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} - \left[\mathcal{Y} + \alpha^T\bar{\lambda}\bar{\lambda}^T + Cov_s\left(\lambda, \left(\lambda^T\alpha\right)\right) \right] \mu$$

where[4]

$$\mathcal{Y}_w(k,h) = \sum_l s_l \Phi_l \left(\left((I-\Omega)^{-1}\alpha\right)_{(:,k)}, \left((I-\Omega)^{-1}\alpha\right)_{(:,h)} \right) + \sum_i \lambda_i^W \Phi_i \left(\left(\widehat{(I-\Omega)^{-1}\alpha}\right)_{(:,k)}, \left(\widehat{(I-\Omega)^{-1}\alpha}\right)_{(:,h)} \right)$$

[4]The hatted matrices $\widehat{(I-\Omega)^{-1}}$ and $\widehat{(I-\Omega)^{-1}}\alpha$ are defined as

$$\widehat{(I-\Omega)^{-1}} \equiv \begin{pmatrix} (I-\Omega)^{-1} \\ \mathbb{O}^{H+J\times N} \end{pmatrix}$$

$$\widehat{(I-\Omega)^{-1}}\alpha \equiv \begin{pmatrix} (I-\Omega)^{-1}\alpha \\ I_{H+J} \end{pmatrix}$$

$$\mathcal{Y}(k,i)=\sum_l s_l\Phi_l\left((I-\Omega)^{-1}_{(:,i)},\left((I-\Omega)^{-1}\alpha\right)_{(:,k)}\right)+\sum_t\lambda_t^W\Phi_t\left(\widehat{(I-\Omega)}^{-1}_{(:,i)},\left(\widehat{(I-\Omega)^{-1}}\alpha\right)_{(:,k)}\right)$$

and the substitution operators Φ are defined as

$$\Phi_l\left(X,Y\right)=\frac{1}{2}\sum_i\sum_{j\neq i}\beta_{il}\tilde{\beta}_{jl}\sigma_{ij}^l\left(X_i-X_j\right)\left(Y_i-Y_j\right)$$

$$\Phi_i\left(X,Y\right)=\frac{1}{2}\sum_j\sum_{t\neq j}\omega_{ij}\tilde{\omega}_{it}\theta_{jt}^i\left(X_j-X_t\right)\left(Y_j-Y_t\right)+$$

$$+\frac{1}{2}\sum_j\sum_k\omega_{ij}\tilde{\alpha}_{ik}\theta_{jk}^{iL}\left(X_j-X_{N+k}\right)\left(Y_j-Y_{N+k}\right)+\frac{1}{2}\sum_j\sum_k\tilde{\omega}_{ij}\alpha_{ik}\theta_{jk}^{iL}\left(X_j-X_{N+k}\right)\left(Y_j-Y_{N+k}\right)$$

$$+\frac{1}{2}\sum_k\sum_{h\neq k}\alpha_{ik}\tilde{\alpha}_{ih}\theta_{kh}^{iL}\left(X_{N+k}-X_{N+h}\right)\left(Y_{N+k}-Y_{N+h}\right)$$

Combining Lemma 3 with equation (19) yields the relative demand block:

$$\mathbb{D}_L l=-\mathbb{D}_w\begin{pmatrix}\pi_w\\ \pi_r\end{pmatrix}+\mathbb{D}_G G \tag{20}$$

where

$$\mathbb{D}_L\equiv\mathcal{S}^{-1}\left(\begin{pmatrix}I_H\\ \mathbb{O}_J\end{pmatrix}-\alpha^T\lambda\right)\mathcal{S}_L$$

$$\mathbb{D}_w\equiv\mathcal{S}^{-1}\left[\alpha^T\lambda\begin{pmatrix}I_K & \mathcal{Z}\end{pmatrix}\mathcal{S}\left(I-\mathbf{1}\varsigma^T\right)+Cov_s\left(\lambda^T\alpha\right)+\mathcal{Y}_w\right]+$$

$$+\mathcal{S}^{-1}\left[\alpha^T\lambda\left(\Theta^T-s\mathbf{1}^T\right)diag\left(\bar{\lambda}\right)-Cov_s\left(\lambda,\left(\lambda^T\alpha\right)\right)-\mathcal{Y}\right]\left(I-\Delta\right)\left(I-\Omega\Delta\right)^{-1}\alpha$$

$$\mathbb{D}_G\equiv\mathcal{S}^{-1}\left(I-\alpha^T\lambda\begin{pmatrix}I_K & \mathcal{Z}\end{pmatrix}\right)\left[\left(I-\Omega\right)^{-1}\alpha\right]^T$$

Equation (19) determines relative factor prices, while the aggregate demand equation (15) pins down the price level. Under Assumption 1, equation (20) is a system of rank $H+J-1$. Both sides vanish when pre-multiplied by the vector ς^T, consistent with the fact that factor income shares are constrained to sum to 1 (so that $\varsigma^T d\log\varsigma=0$).

5 Money and Spending multipliers

Section 5.1 characterizes the equilibrium of the model, by combining the supply and demand blocks derived in Section 4. This allows us to derive the response of factor prices and employment to changes in money supply and government spending. Building on this analysis, Section 5.2 provides two sets of assumptions under which the spending multiplier is the same as in a representative agent economy. In the first, spending replicates the aggregate consumption basket, thereby leaving relative factor demand unaffected. In the second, all agents face the same nominal and real rigidities. Section 5.3 then illustrates what happens when these assumptions are violated, isolating the role of different nominal and real rigidities.

5.1 Wage and employment multipliers

The supply and demand blocks from Section 4 are summarized by three equations:

$$\begin{cases} l = \mathbb{S}_w \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \mathbb{S}_G G & \text{supply} \\ \delta_{\bar{\beta}}(\alpha) \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \varsigma_L^T l = d \log M & \text{aggregate demand} \\ \mathbb{D}_L l = \mathbb{D}_w \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} + \mathbb{D}_G G & \text{relative demand} \end{cases}$$

To characterize wage and employment multipliers, one needs to intersect supply and demand. Substituting for employment from the supply block allows to relate factor prices with money supply and government spending:

$$\begin{cases} \left[\delta_{\bar{\beta}}(\alpha) + \varsigma_L^T \mathbb{S}_w\right] \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = d \log M - \varsigma_L^T \mathbb{S}_G G & \text{aggregate} \\ \left[\mathbb{D}_L \mathbb{S}_w + \mathbb{D}_w\right] \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = \left[\mathbb{D}_G - \mathbb{D}_L \mathbb{S}_G\right] G & \text{relative} \end{cases} \tag{21}$$

The system (21) highlights that changes in the money supply have an impulse effect only on the aggregate demand block, while changes in government spending have a direct effect on the relative demand for different factors. The distinction is even starker when there are no wealth effects in labor supply. In this case we have $\mathbb{S}_G = \mathbb{O}$, and government spending directly affects only the relative demand block.

The system (21) allows us to provide a more explicit characterization of factor prices. Even though the matrix $\mathbb{D}_L \mathbb{S}_w + \mathbb{D}_w$ is not invertible,[5] under Assumption 1 we are able to characterize the factor price changes which restore the equilibrium in all factor markets after a monetary or a spending shock. This is done in Lemma 4

Lemma 4. *Denote by $\mathcal{D} \equiv \mathbb{D}_L \mathbb{S}_w + \mathbb{D}_w$ and consider the decomposition*

$$\mathcal{D} = \mathcal{D}_{XS}\left(I - \mathbf{1}\varsigma^T\right) - \bar{\mathcal{D}} s^T$$

[5] Because $\varsigma^T \mathbb{D}_L = \varsigma^T \mathbb{D}_w = 0$ (see Section 4)

where $\bar{\mathcal{D}} \equiv \mathcal{D}\mathbf{1}$*. Under Assumption 1 the matrix* $\mathcal{D}_{XS}$ *is invertible. The unique factor price change which preserves the equilibrium in all factor markets is given by* $\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} \propto \mathbf{1} + \mathcal{D}_{XS}^{-1}\mathcal{D}$*. Formally,*

$$\mathcal{D}\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} \propto \mathbf{1} + \mathcal{D}_{XS}^{-1}\mathcal{D}$$

The unique relative price change which restores the equilibrium in all factor markets after a change in government spending is given by $\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = \mathcal{D}_{XS}^{-1}\left[\mathbb{D}_G - \mathbb{D}_L\mathbb{S}_G\right]G$*. Formally,*

$$\mathcal{D}\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = \left[\mathbb{D}_G - \mathbb{D}_L\mathbb{S}_G\right]G,\ \varsigma^T\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = 0 \Longleftrightarrow$$

$$\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = \mathcal{D}_{XS}^{-1}\left[\mathbb{D}_G - \mathbb{D}_L\mathbb{S}_G\right]G$$

Lemma 4 allows us to derive an explicit expression for money and spending multipliers. This is done in Sections 5.1.1 and 5.1.2.

5.1.1 Money multipliers

Proposition 1 builds on Lemma 4 to derive the money multiplier, $\mathbb{L}_m$. This multiplier captures the response of each agent's employment to a 1% change in money supply. Therefore $\mathbb{L}_m$ is a vector with H components, corresponding to each agent.

Proposition 1. *The money multiplier is given by*

$$\mathbb{L}_m = \frac{\mathbb{S}_w\left[\mathbf{1} + \mathcal{D}_{XS}^{-1}\bar{\mathcal{D}}\right]}{\mathcal{E}^T\left[\mathbf{1} + \mathcal{D}_{XS}^{-1}\mathcal{D}\right]} \tag{22}$$

where we denoted by

$$\mathcal{E}^T \equiv \delta_{\bar{\beta}}(\alpha) + \varsigma_L^T\mathbb{S}_w$$

To understand equation (22) we need to trace back the effect of a monetary expansion on factor prices. From the equilibrium system (21) we read that changes in money supply have no direct effect on the relative demand for different factors. The change in factor prices induced by monetary policy therefore must preserve the equilibrium in all factor markets. From Lemma 4 we then know that factor prices must change proportionately to the vector $\mathbf{1} + \mathcal{D}_{XS}^{-1}\mathcal{D}$.

Intuitively, every proportional increase in factor prices (as captured by the vector $\mathbf{1}$) generates an imbalance between the demand and supply of different factors, given by the vector $\bar{\mathcal{D}}$. Demand increases for workers with stickier wages, or who are employed by stickier sectors (etc.), because their products become relatively cheaper. By contrast, supply increases for workers with higher Frish elasticity $\frac{1}{\varphi}$. Moreover, the relative price of goods which rely less on fixed factors falls, thereby increasing demand for the factors which produce them. These imbalances must be compensated by a relative price adjustment, which is given by $\mathcal{D}_{XS}^{-1}\bar{\mathcal{D}}$.

While the change in factor prices must be proportional to $\mathbf{1}+\mathcal{D}_{XS}^{-1}\mathcal{D}$, its size is determined by imposing that aggregate GDP equals the money supply (equation (21)). This yields

$$\begin{pmatrix} \pi_w \\ \pi_r \end{pmatrix} = \frac{\mathbf{1}+\mathcal{D}_{XS}^{-1}\bar{\mathcal{D}}}{\mathcal{E}^T\left[\mathbf{1}+\mathcal{D}_{XS}^{-1}\mathcal{D}\right]} d\log M \tag{23}$$

The denominator in (22) and (23) corresponds to the change in aggregate expenditure when factor prices change by $\mathbf{1}+\mathcal{D}_{XS}^{-1}\mathcal{D}$. Expenditure is given by aggregate output times aggregate prices. The former equals aggregate employment, which changes by $\varsigma_L^T\mathbb{S}_w\left[\mathbf{1}+\mathcal{D}_{XS}^{-1}\bar{\mathcal{D}}\right]$, while the latter is given by the change in factor prices, $\mathbf{1}+\mathcal{D}_{XS}^{-1}\mathcal{D}$, times its pass-through into aggregate consumer prices, $\delta_{\bar{\beta}}(\alpha)$. The employment multiplier is given by the factor price multiplier times the labor supply elasticity $\mathbb{S}_w$.

In an economy with a representative agent and no fixed factors, a monetary expansion cannot create any imbalance in relative factor demand. Therefore we have $\bar{\mathcal{D}}=0$. As we derived in Example 2, labor supply changes by $\frac{1-\delta_\beta(\alpha)}{\gamma+\varphi}\pi_w$. Therefore the money multiplier is

$$\mathbb{L}_m = \frac{\frac{1-\delta_\beta(\alpha)}{\gamma+\varphi}}{\delta_{\bar{\beta}}(\alpha)+\frac{1-\delta_\beta(\alpha)}{\gamma+\varphi}}$$

The examples in Section 5.3 below show that, in the presence of heterogeneity, the aggregate money multiplier is larger than in a representative agent economy with the average nominal and real rigidities.

5.1.2 Spending multipliers

Proposition 2 builds on Lemma 4 to derive the spending multiplier, $\mathbb{L}_G$. This multiplier captures the response of each agent's employment to sector level government spending. Therefore $\mathbb{L}_G$ is an $H\times N$ matrix, where the (h,i) component corresponds to the effect of spending in sector i on agent h's employment.

Remark 5. To derive the spending multiplier it is critical to specify the monetary response to spending. Proposition 2 assumes that the central bank keeps the money supply constant. This is without loss of generality, and multipliers under other policy rules can be obtained as appropriate combinations of $\mathbb{L}_m$ and $\mathbb{L}_G$.

Proposition 2. *Assume that the central bank increases money supply by the same amount as government spending ($d\log M = \mathbf{1}^T G$). Then the spending multiplier is given by*

$$\mathbb{L}_g = \left(I - \mathbb{L}_m \varsigma_L^T\right) \mathbb{S}_G + \left[\mathbb{S}_w - \mathbb{L}_m \mathcal{E}^T\right] \mathcal{D}_{XS}^{-1} \left[\mathbb{D}_G - \mathbb{D}_L \mathbb{S}_G\right] \tag{24}$$

The matrix $\mathbb{S}_G$ in equation (24) reflects the fact that spending increases labor supply through a wealth effect, as derived in Section 4. The term $\mathcal{D}_{XS}^{-1}\left[\mathbb{D}_G - \mathbb{D}_L \mathbb{S}_G\right]$ instead reflects the effect of spending on factor prices. From the equilibrium system (21) we that changes in government spending affect relative factor demand proportionately to the matrix $\mathbb{D}_G - \mathbb{D}_L \mathbb{S}_G$. Factor prices therefore must adjust to restore equilibrium in the factor markets. From Lemma 4, the necessary adjustment is given by $\mathcal{D}_{XS}^{-1}\left[\mathbb{D}_G - \mathbb{D}_L \mathbb{S}_G\right] G$. The employment response is obtained by multiplying factor prices times the labor supply elasticity $\mathbb{S}_w$.

The terms which depend on the money multiplier $\mathbb{L}_m$ capture the effect of changes in factor prices and employment on aggregate expenditure, which tightens the cash-in-advance constraint.

In an economy with a representative agent and no fixed factors, a monetary expansion has no effect on relative factor demand: $\mathbb{D}_G - \mathbb{D}_L \mathbb{S}_G = 0$. From Example 2, we have $\mathbb{S}_w = \frac{1-\delta_\beta(\alpha)}{\gamma+\varphi}$ and $\mathbb{S}_G = \frac{\gamma}{\gamma+\varphi}$. Therefore the spending multiplier is

$$\mathbb{L}_G = \left(1 - \mathbb{L}_m\right) \frac{\gamma}{\gamma+\varphi} \tag{25}$$

Propositions 3 and 4 below consider two special cases where the spending multiplier in the heterogeneous agents economy has the same form as in (25). The examples in Section 5.3 below instead illustrate when these conditions are not satisfied, showing that the multiplier is larger when spending increases the demand for agents who face more nominal and less real rigidities.

5.2 Two "as if" results

Propositions 3 and 4 present two special cases where the spending multiplier in the heterogeneous agents economy has the same form as with a representative agent. In the first, government spending replicates the composition of the aggregate consumption basket. In the second instead all agents face the same nominal and real rigidities.

Proposition 3. *Assume that either (i) agents have the same wealth effects in labor supply and Frish elasticity ($\Gamma = \gamma I$, $\Phi = \varphi I$) and there are no fixed factors or (ii) agents have no wealth effects in labor supply. Then if spending replicates the aggregate consumption basket ($G \propto \bar{\beta}$) the spending multiplier is given by*

$$\mathbb{L}_G G = \left(\mathbf{1} - \mathbb{L}_m\right) \frac{\gamma}{\gamma+\varphi} \left(\mathbf{1}^T G\right)$$

Proof. It is easy to verify that $\mathbb{D}_G G = \mathbf{0} \Longleftrightarrow G \propto \bar{\beta}$. This means that government spending does not affect relative factor demand when it replicates the aggregate consumption basket. Under assumption (ii) spending has no direct effect on labor supply, while under assumption (i) the effect on labor supply is given by $\mathbb{S}_G = \frac{\gamma}{\gamma+\varphi}\mathbf{1}\mathbf{1}^T G$. Plugging into (24) yields the result. □

Proposition 4. *The sectoral composition of government spending is irrelevant for aggregate employment in an economy with flexible prices and no fixed factors, and where all agents have the same Frish elasticity ($\Phi = \varphi I$) and wealth effects in labor supply ($\Gamma = \gamma I$).*

Proof. Example 1 derives the response of labor supply to factor prices in a flex-price economy:

$$\mathbb{S}_w = \frac{1-\gamma}{\gamma+\varphi}\left(I - \lambda^T\alpha\right)$$

As shown in the example, changes in nominal wages have no effect on the aggregate labor supply ($s^T\mathbb{S}_w = 0$). Moreover, in this economy the response of labor supply to government spending is given by

$$\mathbb{S}_G = \frac{\gamma}{\gamma+\varphi} S^{-1}\left[\left(I-\Omega\right)^{-1}\alpha\right]^T$$

Following Remark 4, the response of aggregate employment is then given by

$$\bar{l} = \left(1 - \bar{\mathbb{L}}_m\right)\frac{\gamma}{\gamma+\varphi}\left(\mathbf{1}^T G\right)$$

Hence aggregate employment only depends on total spending, not on its composition. □

Proposition 4 considers an economy where all agents face the same nominal and real rigidities. The assumptions that (i) all agents have the same wealth effects and Frish elasticity and (ii) there are no fixed factors rule out heterogeneous real rigidities. The flex-price assumption instead rules out nominal rigidities altogether. Consistent with our previous discussion, the allocation of spending has no effect on aggregate employment in this economy.

5.3 The role of heterogeneity

This Section illustrates the implications of heterogeneous nominal and real rigidities through some concrete examples. Examples 4 and 5 isolate the role of heterogeneous nominal rigidities and heterogeneous labor supply elasticities. Examples 6 and 7 show how the input-output structure determines the degree of nominal rigidity and the demand elasticity faced by each agent. Finally, Examples 8 and 9 illustrate the role of fixed factors.

Example 4. Wage rigidity

Consider and economy with two agents and one final good, as in Figure 1. The two agents supply different types of labor, and face different wage rigidity. We denote their wage adjustment probabilities by $\delta_{sticky} < \delta_{flex}$.

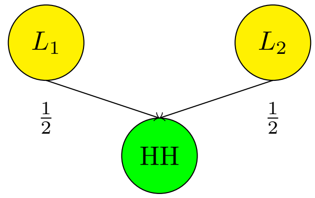

Figure 1:

The two workers have the same initial share in the production of the final good, and their elasticity of substitution is θ. Both agents have the same inverse Frish elasticity, equal to φ, and there are no wealth effects in labor supply ($\Gamma = \mathbb{O}$).

After a monetary expansion, employment increases by more for the worker with stickier wage:

$$\mathbb{L}_m^{sticky} - \mathbb{L}_m^{flex} \propto \frac{\varphi\theta\bar{\bar{\delta}}}{1+\varphi\theta\bar{\bar{\delta}}}(\delta_{flex} - \delta_{sticky})d\log M \tag{26}$$

This is because her labor becomes cheaper, and producers can substitute across the two labor types. Equation (26) shows that the cross-sectional multiplier is largest when workers are perfect substitutes, and it becomes 0 when they are perfect complements.

In the aggregate, the ability to shift demand towards the worker with stickier wages increases the money multiplier. This is given by:

$$\bar{\mathbb{L}}_m = \frac{1 + \frac{(\delta_{flex}-\delta_{sticky})^2}{1-\bar{\delta}}\frac{\varphi\theta}{1+\varphi\theta\bar{\bar{\delta}}}}{1+\varphi\frac{\bar{\delta}}{1-\bar{\delta}} - (\varphi-1)\frac{(\delta_{flex}-\delta_{sticky})^2}{1-\bar{\delta}}\frac{\varphi\theta}{1+\varphi\theta\bar{\bar{\delta}}}}$$

Again, heterogeneity plays no role when workers are perfect complements, and the multiplier is the same as with a representative agent. Instead, the effect of heterogeneity is strongest when workers are perfect substitutes.

In this economy, government spending increases aggregate employment by more when it is directed towards the sticky-wage worker. Specifically, we allow for the government to purchase three goods: labor of the sticky-wage worker, labor of the flex-wage worker, and the final good. With no wealth effects in labor supply, in a representative agent economy government spending fully crowds out private consumption. Therefore the spending multiplier is $\mathbb{L}_G = 0$. With heterogeneous agents, instead, spending on the sticky workers increases aggregate employment, and vice versa:

$$\bar{\mathbb{L}}_g \propto \frac{\delta_{flex} - \delta_{sticky}}{1+\varphi\theta\bar{\bar{\delta}}}(G_{sticky} - G_{flex})$$

Intuitively, spending increases the demand for the worker towards which it is directed, and therefore her relative wage, unless the two workers are perfect substitutes. If spending is directed towards the sticky worker, real wages increase on average, because the increase in the relative wage of this worker has a smaller impact on final prices than the fall in the relative wage of the flex-wage worker. This in turn increases the aggregate labor supply.

Following a similar logic, the change in relative employment induced by directed spending is larger when workers are more complementary:

$$l_h - l_m \propto \frac{G_h - G_m}{1 + \varphi\theta\bar{\delta}}$$

Again, spending increases the demand and wage of the worker towards which it is directed. In response, producers substitute towards the other worker. The demand spillover is smaller when workers are more complementary.

Consistent with proposition 3, spending on the final good has no effect on aggregate employment.

Example 5. Labor supply elasticity

Consider the same economy as in Figure 1. Assume now that the two workers face the same wage rigidity, but have different inverse Frish elasticity ($\varphi_I > \varphi_E$).

In this economy, a monetary expansion benefits more elastic workers:

$$l_E - l_I = (\varphi_I - \varphi_E)\frac{\theta\delta}{1 + \bar{\varphi}\theta\delta}\bar{\mathbb{L}}_m$$

Intuitively, an increase in money supply raises all wages proportionately on impact. However the labor supply of the elastic worker increases by more, thereby bidding down her wage. Producers thus shift demand towards this worker, until the labor markets are in equilibrium. Note that the cross-sectional effect is largest when workers are perfect substitutes, and it disappears when they are perfect complements.

The ability of producers to substitute towards the more elastic workers also raises the aggregate money multiplier. This is given by:

$$\bar{\mathbb{L}}_m = \frac{1}{1 + \bar{\varphi}\frac{\delta}{1-\bar{\delta}} - \frac{\delta}{1-\bar{\delta}}\frac{\theta\delta}{1+\bar{\varphi}\theta\delta}(\varphi_I - \varphi_E)^2}$$

Again, the effect of heterogeneity disappears when production is Leontief, and the multiplier is the same as in a representative agent economy.

Following a similar reasoning as in Example 4, government spending increases aggregate employment if and only if it is directed towards the more elastic worker:

$$\bar{\mathbb{L}}_g \propto \frac{\varphi_I - \varphi_E}{\bar{\varphi} + \varphi_E\varphi_I\theta\delta}(G_E - G_I)$$

This is because spending increases the demand of the worker towards which it is directed, and therefore her relative wage (as long as the two workers are not perfect substitutes). The response of aggregate labor supply is dominated by the more elastic worker, hence it increases if and only if her wage increased.

Finally, the effect of directed spending on cross-sectional employment is larger when the two workers are more complementary:

$$l_h - l_m \propto \frac{G_h - G_m}{\bar{\varphi} + \varphi_h \varphi_m \theta \bar{\bar{\delta}}}$$

Example 6. Chain-weighted nominal rigidities

This example illustrates that different price rigidities along the production chain(s) where workers are employed have the same implications as different wage rigidity. In fully general terms, what matters is the differential pass-through of the wage of our two workers into consumer prices $(\delta_{\bar{\beta}}(\alpha_h) - \delta_{\bar{\beta}}(\alpha_m))$. The example makes this argument in the context of a single vertical chain, where one agent produces the intermediate input, while the other agent assembles the final good. The elasticity of substitution between intermediate inputs and labor in the production of final goods is θ. Our economy is depicted in Figure 2.

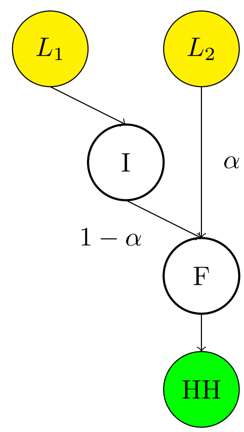

Figure 2:

As in Example 4, we assume that agents have the same Frish elasticity and there are no wealth effects in labor supply. We now assume that wages are flexible, while both the intermediate and final good sectors have sticky prices, with the same adjustment probability δ.

In this economy, the cross-sectional money multiplier is given by

$$l_I - l_F \propto \delta(1-\delta)\frac{\varphi\theta}{1 + \varphi\theta\bar{\bar{\delta}}} d\log M$$

where $\delta(1-\delta)$ is the difference between the wage pass-through into final prices of the final good versus the intermediate good worker. All other multipliers are also the same as in Example 4, simply replacing the difference in wage rigidity with the difference in pass-through.

Example 7. Chain-weighted elasticity of substitution

Now consider an economy with two consumption goods and two workers, as in Figure 5.3.

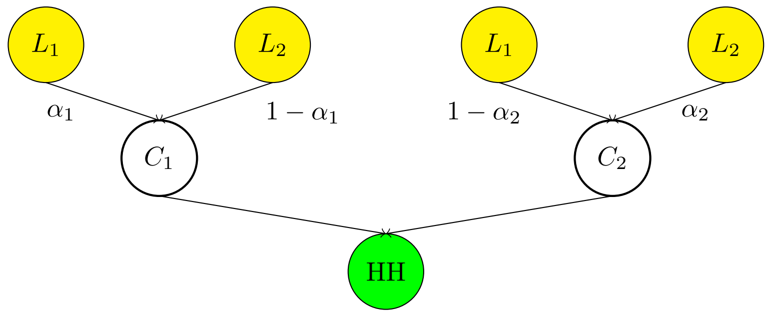

Figure 3:

Both consumption goods use both workers, although in different proportions. Wages are flexible, and final good prices are sticky with the same adjustment probability δ. Workers have the same labor supply elasticity. The substitution elasticity between workers is θ in both sectors, and the elasticity of substitution between final goods is σ.

In this example, both workers are subject to the same nominal and real rigidities. Therefore the aggregate money multiplier is the same as in a representative agent economy, and monetary policy has no redistributive effect. Similarly, directed government spending has no effect on aggregate labor supply. Nonetheless, it is interesting to study the cross-sectional spending multiplier. This is given by:

$$l_2 - l_1 = \frac{\varphi \beta_1 (1-\beta_1)(\alpha_1 - \alpha_2)\left(\frac{G_1}{\beta_1} - \frac{G_2}{1-\beta_1}\right)}{1 + \varphi\left[\frac{\beta_1(1-\beta_1)(\alpha_1-\alpha_2)^2}{s_1(1-s_1)}\sigma\delta + \left(1 - \frac{\beta_1(1-\beta_1)(\alpha_1-\alpha_2)^2}{s_1(1-s_1)}\right)\theta\right]}$$

We see that in this case the relevant elasticity of substitution between the two workers is given by a weighted average between the production and consumption elasticities. Interestingly, the more downstream elasticity is discounted by the price adjustment probability. This is because the effect of a change in the workers' wage on prices is smaller in more downstream sectors.

Example 8. Fixed factors and the labor share

This example illustrates how the presence of a fixed factor affects the response of the economy to monetary policy. Consider an economy with one worker, one fixed factor (which we call capital), and one final good, as in Figure 4. The labor share in the production of final goods is α, while the elasticity of substitution between labor and capital is θ.

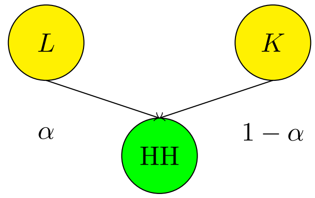

Figure 4:

The employment response to monetary policy is given by

$$\mathbb{L}_m = \frac{1 - \frac{1-\alpha}{1-\alpha+\varphi\theta}}{1 + \varphi\frac{\delta}{1-\delta} - (1+\varphi\theta)\frac{1-\alpha}{1-\alpha+\varphi\theta}}$$

It is easy to verify that the money multiplier is smaller than in an economy with no fixed factors if and only if $\theta < \frac{\delta}{1-\delta}$. This condition is more likely to be satisfied when labor and capital are more complementary, or when prices are more flexible. Intuitively the expansion raises the demand for final goods, thereby bidding up wages and the rental rate of capital. If labor and capital are very complementary asset prices increase much more than wages, because capital is in fixed amount. As a result, real wages increase by less than in an economy with no capital, and the labor share falls. By contrast, if producers can substitute away from capital and towards labor, real wages increase by more than in an economy with no fixed factors, and the labor share also increases.

Example 9. Housing markets

Consider an economy with two worker types (the residents of New York and Boise) and two fixed factors (land in New York and land in Boise). There are two final goods (housing in New York and Boise), which use as inputs local labor and land. The elasticity of substitution between the two inputs is θ. Land in New York is scarcer than in Boise, so that in equilibrium the labor share in New York housing is smaller ($\alpha_{NY} < \alpha_B$). Workers can move between the two cities. The degree of mobility is captured in reduced form by the elasticity of substitution between housing in New York and Boise in final consumption, which we denote by σ. To keep things simple, we assume that population is evenly split between the two cities in the initial equilibrium. The economy is depicted in Figure 5.

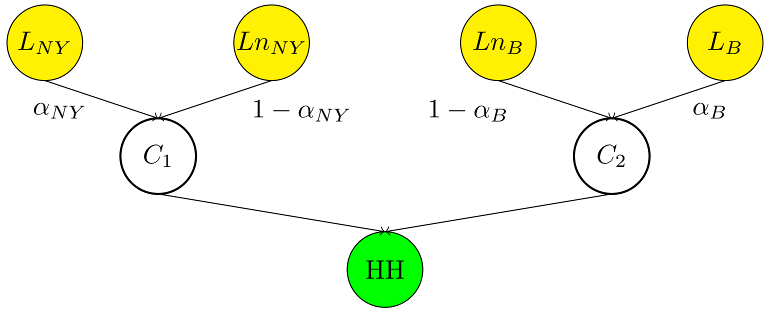

Figure 5:

Let's consider the effect of a monetary expansion on land prices and employment in the two cities. The cross-sectional money multiplier is:

$$l_B - l_{NY} \propto \theta(\sigma\delta - \theta)(\alpha_B - \alpha_{NY}) d\log M \tag{27}$$

Equation (27) tells us that migration elasticities are key in determining the redistributive effects of monetary policy. If people are unwilling to migrate ($\sigma\delta < \theta$), then employment in New York increases more than in Boise. This is because substitution towards labor and away from land in New York is stronger than substitution between New York and Boise. In other words, people prefer to improve their small apartment in New York rather than moving to a bigger one in Boise. If instead workers do not have strong location preferences (for example, they can work remotely), then Boise benefits more from the monetary expansion.

Suppose now that the government decides to invest in construction projects, with the objective to increase aggregate employment. Where should it locate its new constructions? Following the same logic as in Example 4, the projects should be located in Boise if and only if the spending multiplier is increasing in the labor share, that is, if $\theta < \frac{\delta}{1-\delta}$. Formally, the aggregate spending multiplier is given by:

$$\bar{\mathbb{L}}_G \propto \varphi\theta\left(\frac{\delta}{1-\delta} - \theta\right)(\alpha_B - \alpha_{NY})(G_B - G_{NY})$$

6 Quantitative analysis

This section calibrates the model to the US economy, to quantify the redistributive effects of monetary policy and government spending, and the importance of heterogeneity for aggregate multipliers.

We distinguish workers by their income-education group. Individuals in different groups consume different goods, are employed by different sectors, and have different frequencies of wage adjustment and labor supply elasticities. In turn, sectors have different position in the input-output network and hire different bundles of workers and fixed factors.

Section 6.2 studies which income-education groups benefit the most from a monetary expansion, and quantifies by how much heterogeneity increases monetary non-neutrality. Section 6.3 instead studies which groups benefit the most from government spending, when the composition of federal and local spending is as reported in the BEA input-output tables.

6.1 Data

6.1.1 Input-output matrix

The input-output matrix Ω is calibrated based on the input-output tables provided by the BEA.

6.1.2 Consumption shares

The matrix of consumption shares β is calibrated combining data from the BEA input-output tables and the CEX. CEX data allows to infer the share of each income-education group in the consumption of different goods, and each good's share in the aggregate consumption basket is then computed from the national input-output tables.

6.1.3 Employment shares

I calibrate the share of workers in different income-education brackets in each industry's labor costs using data from the American Community Survey. This provides a counterpart to the matrix α of direct factor shares.

6.1.4 Price adjustment probabilities

I calibrate the matrix Δ of sectoral frequencies of price adjustment using data collected by Pasten et al. (2019).

6.2 The money multiplier

6.3 The spending multiplier

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